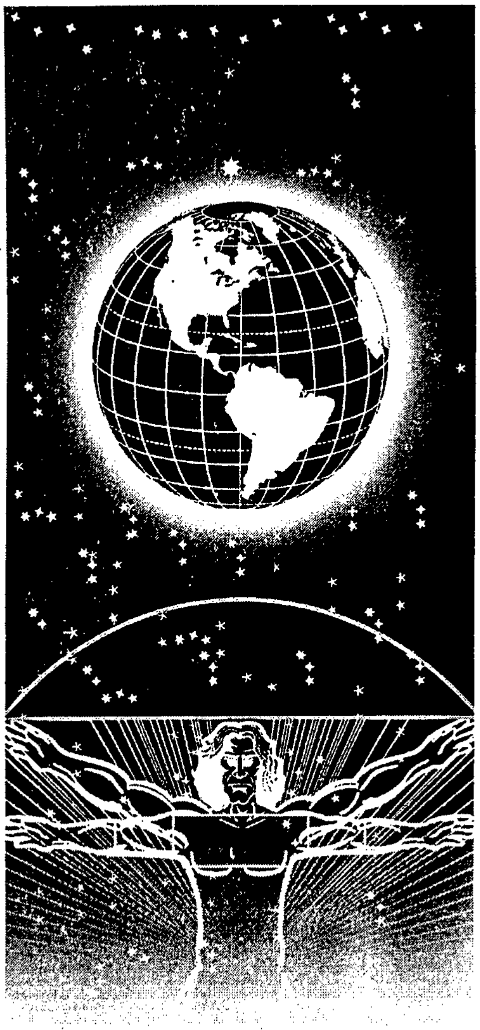
AFRL-HE-BR-TR-1999-0149



UNITED STATES AIR FORCE RESEARCH LABORATORY

WIDEBAND PULSE PROPAGATION IN LINEAR DISPERSIVE BIO-DIELECTRICS USING FOURIER TRANSFORMS

John P. Franzen

AIR FORCE RESEARCH LABORATORY
HUMAN EFFECTIVENESS DIRECTORATE
DIRECTED ENERGY BIOEFFECTS DIVISION
BIOMECHANISMS AND MODELING BRANCH
2503 Gillingham Drive
Brooks AFB, Texas 78235-5102

Approved for public release; distribution unlimited.

February 1999

20000104 042

NOTICES

This report is published in the interest of scientific and technical information exchange and does not constitute approval or disapproval of its ideas or findings.

Using Government drawings, specifications, or other data included in this document for any purpose other than Government-related procurement does not in any way obligate the US Government. The fact that the Government formulated or supplied the drawings, specifications, or other data, does not license the holder or any other person or corporation, or convey any rights or permission to manufacture, use, or sell any patented invention that may relate to them.

The Office of Public Affairs has reviewed this paper, and it is releasable to the National Technical Information Service, where it will be available to the general public, including foreign nationals.

This report has been reviewed and is approved for publication.

JONATHAN KIEL
Project Scientist

RICHARD L. MILLER, Ph.D.
Chief, Directed Energy Bioeffects Division

REPORT DOCUMENTATION PAGE

Form Approved
OMB No. 0704-0188

1. AGENCY USE ONLY *(Leave blank)*

2. REPORT DATE
February1999

3. REPORT TYPE AND DATES COVERED
Final Report July 1995-December 1995

4. TITLE AND SUBTITLE
Wideband Pulse Propagation in Linear Dispersive Bio-dielectrics Using Fourier Transforms

5. FUNDING NUMBERS
PR - 7757
TA - 06
WU - 01

6. AUTHOR(S)
John P. Franzen

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)
Air Force Research Laboratory (AFMC)
Human Effectiveness Directorate
Directed Energy Bioeffects Division
Biomechanisms and Modeling Branch
2503 Gillingham Drive, Building 175E, Brooks AFB, TX 78235-5102

8. PERFORMING ORGANIZATION REPORT NUMBER
AFRL-HE-BR-TR-1999- 0149

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)

10. SPONSORING/MONITORING AGENCY REPORT NUMBER

11. SUPPLEMENTARY NOTES

12a. DISTRIBUTION AVAILABILITY STATEMENT

Approved for public release; distribution unlimited

12b. DISTRIBUTION CODE

13. ABSTRACT *(Maximum 200 words)*

We describe calculations of a plane-wave pulse train of finite duration in a linear lossy dispersive dielectric half-space by means of Fourier integral transforms. The polarization response of Debye and Lorentz media to a finite number of sinewave pulses and the response of a Lorentz medium to a single Gaussian modulated sinewave pulse are given. These solutions are formulated for the case of normal incidence to the planar boundary of a dielectric half-space. Our results show that the Fourier transform accurately reproduces precursor phenomena previously observed for infinitely periodic pulse trains using the Fourier series method extending the analysis to aperiodic waveforms. These results can be computed to arbitrary accuracy because the Fourier integral representation of the propagated pulse is exact, i.e. no analytical or physical approximations are used to arrive at the time-domain solution. We believe that this numerical tool will be useful in developing physical intuition about the dynamics of pulse propagation in bio-tissue which is now made possible by the availability of high-speed computer hardware.

14. SUBJECT TERMS

pulse propagation	Debye media
dispersive dielectric	Lorentz media
Fourier integral	precursor

15. NUMBER OF PAGES
67

16. PRICE CODE

17. SECURITY CLASSIFICATION OF REPORT	18. SECURITY CLASSIFICATION OF THIS PAGE	19. SECURITY CLASSIFICATION OF ABSTRACT	20. LIMITATION OF ABSTRACT
UNCLASSIFIED	UNCLASSIFIED	UNCLASSIFIED	UL

Abstract

We describe calculations of a plane-wave pulse train of finite duration in a linear lossy dispersive dielectric half-space by means of Fourier integral transforms. The polarization response of Debye and Lorentz media to a finite number of sinewave pulses and the response of a Lorentz medium to a single Gaussian modulated sinewave pulse are given. These solutions are formulated for the case of normal incidence to the planar boundary of a dielectric half-space. Our results show that the Fourier transform accurately reproduces precursor phenomena previously observed for infinitely periodic pulse trains using the Fourier series method while extending the analysis to *aperiodic* waveforms. These results can be computed to arbitrary accuracy because the Fourier integral representation of the propagated pulse is exact, i.e. no analytical or physical approximations are used to arrive at the time-domain solution. We believe that this numerical tool will be useful in developing physical intuition about the dynamics of pulse propagation in bio-tissue which is now made possible by the availability of high-speed computer hardware.

Introduction

The propagation of wideband electromagnetic pulses in biological tissues has important health and safety ramifications. A theoretical understanding of the dynamics of transient pulse propagation in dispersive biological materials is essential to the development of exposure guidelines for non-ionizing electromagnetic radiation. Our goal is to develop a computational method for wideband pulse propagation. The method we use is based on the Fourier transform that allows us to investigate the dynamical evolution of electromagnetic pulses in linear, homogeneous, dispersive media by conducting numerical experiments. Previous investigators have used a Fourier series calculation to successfully predict the occurrence of so-called Brillouin precursors in a water half-space for incident pulses having sufficiently short rise times [1]. This report documents the computation of propagated waveforms in linear dispersive half-spaces characterized by Debye and Lorentz permittivity models. These models were chosen to demonstrate the Fourier transform method of calculating the transient response of bio-tissues to both microwave and optical pulsed signals. The essential difference between the Fourier series and the Fourier transform is that the spectrum of the latter is continuous and hence the synthesis of the aperiodic time-domain signal from its spectrum is accomplished by means of integration instead of summation. The specific aperiodic waveforms addressed in this report are square-wave modulated sinusoids of finite duration and the single Gaussian modulated sinewave pulse. Analytical and numerical aspects of the computations are discussed.

Formulation

When the frequency response of the medium is available from knowledge of its transfer function, then pulse propagation is straightforward using the Fourier transform method. We decompose the transmitted pulse into harmonic waves, because harmonic waves have the unique property of retaining their shape even while propagating through highly dispersive media. However, the amplitude of the harmonic wave may decrease due to

absorption by the medium. It is only because harmonic waves do not change shape that the wave speed is precisely defined. For this reason, our discussion of waveforms propagating in dispersive media will be carried out by resolving the waveform into a superposition of harmonic waves.

In a *dispersive* biological medium, each sinusoid of frequency ω, moves at its own speed and has its own characteristic absorption constant. Therefore, our strategy for pulse propagation will be to resolve the incident pulse shape into a superposition of harmonic waves. Propagate each wave of frequency ω for a fixed amount of time at its own phase velocity and then resynthesize the transmitted pulse by adding up all component harmonic waves by means of the inverse Fourier integral while accounting for the attenuation of each component sinusoid over the distance traveled.

Frequency Spectrum

We begin with a review of fundamental theory. Let the time-history of a pulse be designated *f(t)*. Its Fourier spectral density $\tilde{F}(\omega)$ is given by

$$\tilde{F}(\omega) = \int_{-\infty}^{+\infty} f(t) e^{-i\omega t} dt \qquad \textbf{(1)}$$

The Dirichlet conditions[1] are sufficient to ensure the existence of this transform for all physically realizable waveforms, *f(t)*. The Fourier spectrum $\tilde{F}(\omega)$ is a complex number used to describe the amplitude and phase of a sinusoid at angular frequency ω. The complex amplitude of the Fourier component at angular frequency ω is $\tilde{F}(\omega)d\omega$; the area under the Fourier spectrum curve in the interval $d\omega$. This area has the same units as the time-history of the pulsed electric field, i.e. Volts/m. Therefore, the dimensional units of the Fourier transform are different from those of the signal waveform which is being transformed. Specifically, they are the units of the signal multiplied by time, i.e. (Volts/m)/Hz or Volts/m-sec. The quantity $\tilde{F}(\omega)e^{i\omega t}d\omega$ is the sinusoidal variation at angular frequency ω. The time-history of the pulse waveform can be recovered by integrating its Fourier spectral density function or, in other words, evaluating the inverse Fourier transform

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \tilde{F}(\omega) e^{i\omega t} d\omega . \qquad \textbf{(2)}$$

Eqs. 1 & 2 define the direct and inverse Fourier transforms respectively.

[1] 1. The waveform *f(t)* has a finite number of finite discontinuities, has a finite number of maxima and minima, and is absolutely integrable, i.e.

$$\int_{-\infty}^{\infty} |f(t)| dt < \infty.$$

Transfer Function

The transfer function of a linear time-invariant system is defined as the Fourier transform of the impulse response of the linear system. The transfer function of a linear dispersive medium is the steady-state response per unit sinusoidal input as a function of angular frequency ω, and depth, z. Let $\tilde{E}_x(z,\omega)$ represent a frequency domain x-polarized plane wave propagating in the z-direction. It is well-known that $\tilde{E}_x(z,\omega)$ must satisfy the harmonic equation (one-dimensional Helmholz equation):

$$\left(\frac{d^2}{dz^2}+k_z^2\right)\tilde{E}_x(z,\omega)=0. \tag{3}$$

In the air half-space (incident half-space), the general solution of this differential equation can be written as the sum of the incident and reflected traveling waves

$$\tilde{E}_x(z,\omega)=\tilde{E}_x(0,\omega)\left[e^{-ik_z z}+\tilde{\Gamma}(z,\omega)e^{+ik_z z}\right], \tag{4}$$

where $\tilde{\Gamma}$ is the reflection coefficient of medium. In the bio-tissue half-space (transmission half-space), the solution consists of only one traveling wave

$$\tilde{E}_x(z,\omega)=\tilde{T}(0,\omega)\cdot\tilde{E}_x(0,\omega)e^{-ik_z z}, \tag{5}$$

where $\tilde{T}$ is the transmission coefficient.

If we specify the electric field on the plane $z=0$ to be $E_x(0,t)$ and assume there are no reflected waves since we are propagating into an infinite bio-tissue half-space, then using the inverse Fourier transform, Eq. 2, the space-time solution is

$$E_x(z,t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\tilde{T}(0,\omega)\cdot\tilde{E}_x(0,\omega)e^{i(\omega t-k_z z)}d\omega,\quad z\geq 0. \tag{6}$$

In the following we will make use of the fact that the incident pulse $E_x(0,t)$ and the transmitted pulse $E_x(z,t)$ can be considered as the input and output of a linear system with transfer function, $\tilde{H}(z,\omega)$. Thus, if the incident pulse $E_x(0,t)$, whose Fourier spectrum is $\tilde{E}_x(0,\omega)$ at $z=0$ enters a linear medium then

$$\tilde{E}_x(z,\omega)=\tilde{H}(z,\omega)\cdot\tilde{E}_x(0,\omega), \tag{7}$$

where $\tilde{E}_x(z,\omega)$ is the Fourier spectrum of the transmitted pulse, and $\tilde{H}(z,\omega)$ is the steady-state transfer function of the medium. Eq. 7 describes the transmitted pulse spectrum with respect to depth z, the distance from the planar interface separating the two half-

spaces. Eq. 7 is applicable when the input and output are both zero at time, *t=0*, which will be assumed hereafter.

Eqs. 1,2 & 7 provide a systematic method for finding the time-history of a transmitted pulse in a linear dispersive medium. Starting with the incident pulse $E_x(0,t)$ obtain its Fourier transform from Eq. 1

$$\tilde{E}_x(0,\omega)=\int_{-\infty}^{+\infty}E_x(0,t)e^{-i\omega t}dt. \tag{8}$$

Next, use Eq. 7 to determine $\tilde{E}_x(z,\omega)$ from which $E_x(z,t)$ is obtained using the inverse Fourier transform given by Eq. 2

$$E_x(z,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{E}_x(z,\omega)e^{i\omega t}d\omega. \tag{9}$$

Combining Eqs. 7-9 yields an expression for the transmitted pulse at depth z and time t

$$E_x(z,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{H}(z,\omega)\tilde{E}_x(0,\omega)e^{i\omega t}d\omega, \tag{10}$$

$$\text{where: } \tilde{H}(z,\omega)=\tilde{T}(0,\omega)e^{-i\tilde{k}z}. \tag{11}$$

The Fresnel transmission coefficient, $\tilde{T}(0,\omega)$, in Eq. 11 accounts for the effect on the amplitude and phase of the transmitted pulse $E_x(z,t)$ due to the planar boundary at $z=0$ separating the two half-spaces. It is usually expressed in terms of the indices of refraction of the two half-spaces. Assuming the incident half-space to be air with index of refraction nearly equal to unity, the Fresnel transmission coefficient at normal incidence is

$$\tilde{T}(\omega)=\frac{2}{1+\tilde{n}(\omega)} \text{ and } \tilde{n}(\omega)=\sqrt{\tilde{\varepsilon}(\omega)}, \tag{12}$$

where $\tilde{n}(\omega)$ is the complex (frequency dependent) index of refraction of the half-space containing the transmitted pulse and $\tilde{\varepsilon}(\omega)$ is the complex (frequency dependent) relative permittivity of the medium.

The complex exponential factor: $\exp\{-i(\tilde{k}z-\omega t)\}$ in Eq. 11 effects the propagation in time and space of the waveform $E_x(z,t)$ by applying the proper phase shift and attenuation to each sinusoidal component of $\tilde{E}_x(z,\omega)$ to produce a traveling wave. In general, the wavenumber $\tilde{k}(\omega)=\omega/\tilde{v}(\omega)$ and phase velocity are both complex and frequency dependent. The real part of the wavenumber describes the phase shift per unit distance traveled by each sinusoidal component while the imaginary part of the

wavenumber accounts for the exponential amplitude attenuation per unit distance traveled by each sinusoid. The frequency response of a dispersive medium is determined by the frequency dependence of the complex permittivity of the medium and its relationship to the wavenumber at frequency ω. For bio-tissue, with relative magnetic permeability of unity, we have

$$\tilde{k}(\omega) = (\omega / c)\sqrt{\tilde{\varepsilon}(\omega)}, \qquad \textbf{(13)}$$

where c is the free-space velocity of light. The wavenumber (or equivalently the relative permittivity) is, in turn, dependent on the medium in which the wave propagates.

To simplify the pulse propagation calculations in this report we will make the following assumptions: 1.) all materials are nonmagnetic, 2.) the dielectric materials (bio-tissue) are isotropic and homogeneous, 3.) plane waves are normally incident on a planar interface separating air and bio-tissue, and 4.) the single planar interface separating the two half-spaces is infinite in extent so that diffraction may be ignored. For high frequency wideband pulses propagating in dispersive tissue, we believe these assumptions will still yield computational results accurate to first order and have heuristic value.

Debye Medium

Liquid water is a principle constituent of all living organisms. Protoplasm, the basic material of living cells, consists of a solution in water of fats, carbohydrates, proteins, salts. Water acts as a solvent, transporting, combining, and chemically breaking down these substances. Water plays a key role in the metabolic breakdown of proteins and carbohydrates. The human body is 65% water by weight; some tissues such as the brain and lung are nearly 80% water. It is well known that the dielectric properties of tissue are primarily determined by tissue water content with both permittivity and conductivity increasing with increases in the concentration of tissue water at RF and microwave frequencies.

At the microscopic scale, the bonding of the hydrogen atoms to the highly electronegative oxygen atom makes water a polar molecule. The triangular geometry of the water molecule together with its electrical polarity produce strong intermolecular bonds that account for water's unique combination of physical properties: a high boiling point, a solid phase that is less dense than liquid phase (ice floats!), high specific heat, excellent solvent properties, and high dielectric permittivity.

The triangular structure of the water molecule which gives rise to a permanent dipole moment is the source of the polarization response of water. This dipole moment causes the water molecules to resist the random orientation due to thermal agitation and align themselves with the direction of an impressed microwave field. (The applied electric field, in fact, biases the orientation of the molecules to only a small extent, in comparison to the thermal motion of the water molecules [7].) An a.c. microwave field will produce a sinusoidal torque that will cause individual water molecules or molecular groups to rotate with an angular velocity proportional to the torque. The angular momentum of the water

molecules results in friction with neighboring molecules and converts thereby to linear momentum, which by definition is heat in liquids and gases. This response of the water molecules is called relaxation polarization and is dependent on the frequency of the electric field. P. Debye [6] was the first to successfully model the relaxation polarization response of water as due to the rotational motion of the water molecule in a damping frictional medium. See, for example, references [7,8] for detailed accounts of modern theories of the macroscopic dielectric response of water given in terms of molecular quantities.

The Debye equation has been used to model the frequency-dependent polarization response of water and ice to an impressed microwave field over the frequency range 0.3 to 300 GHz. The complex relative permittivity of water modeled by the Debye equation is

$$\tilde{\varepsilon}(\omega) = \varepsilon_\infty + \frac{\varepsilon_s - \varepsilon_\infty}{1 + i\omega\tau} - \frac{i\sigma}{\omega\varepsilon_0}, \tag{14}$$

where τ is the dielectric relaxation time, and ω is the angular frequency of the impressed time-varying field, and $\varepsilon_0 = 10^{-9}/36\pi\ (\mathrm{F/m})$ is the permittivity of free-space. The relaxation constant, $\tau = 8.1 \times 10^{-12}\ \mathrm{s}$, measures the decay time of the macroscopic polarization of water when the external field is removed. It corresponds to the water absorption peak located near 20 GHz from the relation $\omega\tau=1$. The constants $\varepsilon_\infty = 5.5$ and $\varepsilon_s = 78.2$ are the limiting values of the effective dielectric constants of water at the high and low frequencies, respectively. The ionic conductivity of pure water at 25 °C is represented by the constant $\sigma = 10^{-4}$ Siemans / m. These numerical values are temperature dependent and were obtained by empirical curve fitting methods applied to dielectric measurements [2]. (Note: the departure of ε_∞, from unity is due to the fact that only a limited amount of data spanning a finite range of frequencies were in used to estimate the parameters in the Debye equation.) These parameter values will be used in the following computations for water based tissue described as a single relaxation time-constant Debye medium.

Lorentz Medium

Another model for tissue absorption and dispersion in the range 3×10^{14} to 3×10^{15} Hz (i.e. in the optical range from the u.v. (0.1 μm) to the near i.r. (1 μm)) comes from the mechanical model of electron resonance developed by Lorentz and Drude. This model of electronic resonance is not exact, since it ignores the fact that the electron's motion follows the laws of quantum mechanics rather than newtonion mechanics. Notwithstanding the known limitations inherent in the mechanical analogy, this model gives good results [5]. This dielectric model describes the resonance polarization in a homogeneous, isotropic dielectric characterized by a single resonance frequency ω_0 for which the complex relative permittivity is given by

$$\tilde{\varepsilon}(\omega) = 1 - \frac{b^2}{\omega^2 - \omega_0{}^2 + 2i\delta\omega}. \tag{15}$$

Here, b^2 is the square of the plasma frequency of the medium, δ is a damping constant to account for the energy lost by the harmonically[2] bound electron at the resonant frequency ω_0. The damping that occurs is not due to the electron moving in a viscous fluid, but is due to the energy lost by the electron by radiation as the result of forced oscillations. The medium parameters: $\omega_0 = 4\times10^{16}\,/\,\mathrm{s}$, $b^2 = 20\times10^{32}\,/\,\mathrm{s}^2$, and $\delta = 0.28\times10^{16}\,/\,\mathrm{s}$ describe a strongly dispersive and absorptive medium with an absorption band in the ultraviolet region of the electromagnetic spectrum. These medium parameters were suggested in [3] and are the same numerical values used by Brillouin in [9,10]. They will be used in the following computations for the single resonance Lorentz medium.

Pulse Types

It remains to define the two types of transient waveforms to be discussed. The specific waveforms addressed in this report are (1) the finite train of pulsed sinewaves and (2) the single Gaussian modulated sinewave pulse. These two waveforms are prototypical, if somewhat idealized, of those generated by real pulsed radio frequency and laser emitters. Perhaps more importantly for this discussion, their Fourier spectra can be found analytically.

Sinewave Pulse

With the aid of the Heaviside unit step function, *H(t)*,

$$H(t-a) = \begin{cases} 0, & t < a \\ 1, & t \geq a \end{cases}, \tag{16}$$

we can write a concise analytical expression for any finite pulse train.

We consider first a single pulse of sinewave carrier ω_c amplitude A, duration τ at $z=0$ as illustrated in Figure 1. This can be written as

$$f(t) = A\cdot\sin(\omega_c t)\left[H\left(t+\frac{\tau}{2}\right) - H\left(t-\frac{\tau}{2}\right)\right]. \tag{17}$$

Its Fourier transform is pure imaginary, is derived in Appendix-A, (see Eq. A-9) and can be written as follows:

$$\tilde{F}(\omega) = -\frac{iA\tau}{2}\left(\mathrm{sinc}(\alpha) - \mathrm{sinc}(\beta)\right), \tag{18}$$

where $i = \sqrt{-1}$; $\mathrm{sinc}(x) = \sin x / x$; $\alpha = (\omega_c - \omega)\tau/2$; $\beta = (\omega_c + \omega)\tau/2$.

---

[2] the restoring force is linearly proportional to displacement

The spectrum of this sinewave pulse is similar to the spectrum of its rectangular envelope but translated in frequency by an amount equal to the carrier, ω_c. The width of the main frequency lobe increases as the length of the rectangular modulation waveform, τ, decreases and is independent of the carrier frequency. The first sinc term accounts for most of the energy located near the carrier frequency, while the second sinc term becomes significant only at frequencies far removed from the carrier. It follows also, that the value of $|F(\omega)|$ at $\omega = \omega_c$ is $|F(\omega)| \approx A\tau / 2$. In Figure 2 the magnitude of the spectral density function of a single sinewave pulse of duration τ is plotted over the positive frequency axis. We see that the mainlobe peak is centered at the carrier frequency ω_c and has spectral width of $\Delta\omega = 2/\tau$. It is interesting to note that when this pulse contains an integer number, N, of carrier cycles, i.e. $\tau = 2\pi N / \omega_c$, then the number of zero-crossings in the spectral amplitude function between zero and ω_c is also equal to N. This is true because $\mathrm{sinc}(\alpha) = \mathrm{sinc}[\pi N(1 - \omega/\omega_c)] = 0$; when $\omega/\omega_c = 1/N, 2/N, \ldots, (N-1)/N$.

We consider next a *finite* number, M, of sinewave pulses, where $M>1$. Without loss of generality, we take the number of pulses in the finite pulse train to be odd: ($M=2N-1$), where the middle pulse is centered at the origin, the pulse width is τ, and the pulse period is t_0. The finite pulse train consisting of an odd number of pulses can be written as follows

$$
\begin{aligned}
f(t) = {} & A\sin(\omega_c t)\sum_{n=1}^{N}\left(H\left[t + nt_0 - \frac{\tau}{2}\right] - H\left[t + nt_0 + \frac{\tau}{2}\right]\right) \\
& + A\sin(\omega_c t)\left(H\left[t + \frac{\tau}{2}\right] - H\left[t - \frac{\tau}{2}\right]\right) \\
& + A\sin(\omega_c t)\sum_{n=1}^{N}\left(H\left[t - nt_0 + \frac{\tau}{2}\right] - H\left[t - nt_0 - \frac{\tau}{2}\right]\right)
\end{aligned}
\qquad \mathbf{(19)}
$$

(Note: Eq. 19 is equivalent to Eq. A-1 in Appendix-A). This finite duration wavetrain is generated by gating or "turning on" each pulse at a positive zero-crossing for an integral number (n_1) of carrier cycles, followed by an integral number (n_2) of "off"-cycles, thereby creating a coherent pulse train with pulse repetition period equal to (n_1+n_2) carrier cycles. A wideband pulse will be created when only a few carrier cycles (n_1) occur during the pulse "on" time. The envelope of the pulse train is amplitude modulated using a rectangular gating function requiring maximum bandwidth in the Fourier spectrum. The Fourier transform of a unit amplitude pulse train is derived in Appendix-A and is shown to be

$$
\begin{aligned}
F[f_r(t)] = & \frac{2i}{t_0}\sum_{n=-\infty}^{n=+\infty}\frac{\sin[(\omega_c - n\omega_0)N\pi/\omega_c]\sin[(\omega+n\omega_0)T/2]}{(\omega_c - n\omega_0)(\omega+n\omega_0)} \\
& -\frac{2i}{t_0}\sum_{n=-\infty}^{n=+\infty}\frac{\sin[(\omega_c + n\omega_0)N\pi/\omega_c]\sin[(\omega+n\omega_0)T/2]}{(\omega_c + n\omega_0)(\omega+n\omega_0)} \\
= & \frac{2i}{t_0}\sum_{n=-\infty}^{n=+\infty}\frac{\sin[(\omega_c - n\omega_0)\tau/2]\sin[(\omega+n\omega_0)T/2]}{(\omega_c - n\omega_0)(\omega+n\omega_0)} \\
& -\frac{2i}{t_0}\sum_{n=-\infty}^{n=+\infty}\frac{\sin[(\omega_c + n\omega_0)\tau/2]\sin[(\omega+n\omega_0)T/2]}{(\omega_c + n\omega_0)(\omega+n\omega_0)}
\end{aligned}
\tag{20}
$$

For a finite duration wavetrain consisting of M pulses, the spectrum will consist of a series of lobes of width $1/Mt_0$ modulated by an envelope which corresponds to the spectrum of one pulse. The separation between lobes is $1/t_0 = PRF$.

Gaussian Pulse

A bell-shaped pulse can be obtained by impressing a Gaussian amplitude modulation on a sinewave carrier. At $t = 0$ this waveform may be expressed as the real part of

$$
f(t) = Ae^{-t^2/(2\tau)^2}e^{i\omega_0 t}, \tag{21}
$$

where $\sqrt{2}\tau$ = standard deviation. Here, A is the peak amplitude, ω_0 is the angular frequency of the sinewave carrier and τ is the pulse width parameter. Since a time span of four standard deviations centered at $t = 0$ contains approximately 95% of the area under the Gaussian curve, a wideband pulse will therefore exist when only a few periods of the sinewave carrier occur during this time span. The Fourier transform of this pulse is derived in Appendix-B.

Numerical Procedures

Positive Frequencies Only

To find the time-history of the propagated waveform for any time $t > 0$, it is necessary to evaluate Eq. 10 by numerical quadrature since a general analytic solution is unavailable. However, we will show that it is only necessary to evaluate the inverse Fourier transform over positive frequencies by using a well-known property of real-valued signals. For real signals, the complex conjugate of Eq. 1 is given by

$$
\tilde{F}^*(\omega) = \int_{-\infty}^{\infty} f(t)e^{i\omega t}dt. \tag{22}
$$

From Eq. 22 follows the Hermitian symmetry of the Fourier transform for real-valued signals so that

$$\tilde{F}(-\omega) = \tilde{F}^*(\omega). \tag{23}$$

This property of the Fourier transform of real-valued signals implies that

$$\left|\tilde{F}(-\omega)\right| = \left|\tilde{F}^*(\omega)\right| = \left|\tilde{F}(\omega)\right|. \tag{24}$$

Whatever shape the modulus of the spectral density function has for positive frequencies, it is the same for negative frequencies, i.e. only one-half of the real signal spectrum is needed to uniquely specify the signal. We divide the integral in Eq. 2 into two parts as

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{0} \tilde{F}(\omega)e^{i\omega t}d\omega + \frac{1}{2\pi}\int_{0}^{\infty} \tilde{F}(\omega)e^{i\omega t}d\omega. \tag{25}$$

By use of Eq. 23 we can write

$$f(t) = \frac{1}{2\pi}\int_{0}^{\infty}\left[\tilde{F}(\omega)e^{i\omega t} + \tilde{F}^*(\omega)e^{-i\omega t}\right]d\omega. \tag{26}$$

We note that the quantity in brackets is real. Therefore, we can rewrite Eq. 26 as

$$2\pi \cdot f(t) = \mathrm{Re}\left\{\int_{0}^{\infty} 2\tilde{F}(\omega)e^{i\omega t}d\omega\right\}. \tag{27}$$

This shows that it is only necessary to perform the numerical quadrature over positive frequencies to recover the propagated real signal.

Error Analysis

In order to determine *f(t)* in Eq. 2 or $E_x(z,t)$ in Eq. 10, integration must be performed along the ω-axis. To perform the integration we choose to uniformly sample the integrand at a finite resolution $\Delta\omega$ over a finite frequency range Ω. We consider the errors resulting from each of these finite approximations and consider the errors due to sampling first.

Aliasing Error

Implementing the numerical quadrature of the inverse Fourier transform by means of the rectangular rule, with fixed step size, $\Delta\omega$, we can approximate Eq. 2 by writing it as an infinite sum over the positive integers which corresponds to folding the integral over positive frequencies as shown in Eq. 27

$$f_s(t) = \frac{\Delta\omega}{\pi}\sum_{n=0}^{\infty}\tilde{F}(n\Delta\omega)e^{in\Delta\omega t}. \tag{28}$$

We recognize Eq. 28 as a complex Fourier series representation of $f_s(t)$. To make this explicit, we invoke Eq. A-12 from Appendix-A and let $\omega_0 = \Delta\omega$ and $F(n\Delta\omega) = c_n t_0 = c_n 2\pi / \Delta\omega$ to obtain

$$f_s(t) = 2\sum_{n=0}^{\infty} c_n e^{in\Delta\omega t}. \tag{29}$$

By uniformly sampling the continuous Fourier spectral density in order to numerically approximate the Fourier integral, we arrive at a Fourier series representation of the function $f_s(t)$ with period $t_0 = 2\pi / \Delta\omega$.

We emphasize that the recovered time function $f_s(t)$ is periodic with repetition period given by $t_0 = 2\pi / \Delta\omega$. This can also be demonstrated by replacing t in Eq. 28 with $t + m \cdot 2\pi / \Delta\omega$ where m is any integer. We obtain

$$\begin{aligned} f_s(t + m \cdot 2\pi / \Delta\omega) &= \frac{\Delta\omega}{\pi}\sum_{n=0}^{\infty} \tilde{F}(n\Delta\omega) e^{in\Delta\omega(t + m \cdot 2\pi/\Delta\omega)} \\ &= \frac{\Delta\omega}{\pi}\sum_{n=0}^{\infty} \tilde{F}(n\Delta\omega) e^{in\Delta\omega t + inm(2\pi)} \\ &= \frac{\Delta\omega}{\pi}\sum_{n=0}^{\infty} \tilde{F}(n\Delta\omega) e^{in\Delta\omega t} \end{aligned} \tag{30}$$

Thus we see that $f_S(t + m \cdot 2\pi / \Delta\omega) = f_S(t)$, i.e. *a function sampled in the frequency domain will be repetitive in the time domain with period,* $t_0 = 2\pi / \Delta\omega$. We must choose $\Delta\omega$ small enough so that t_0 is larger than a time interval which contains most of the energy in the signal, otherwise an unacceptable amount of aliasing error will occur in the reconstructed time domain signal.

Removing the Periodicity

The disadvantage of the Fourier-series representation of Eq. 28 lies in the fact that $f_s(t)$ is a periodic function, so it cannot be used to approximate the aperiodic time domain waveform directly. However, it is possible to retain only that part of $f_s(t)$ inside the interval $0 < t < t_0$ to recover an aperiodic waveform [11]. To accomplish this, we first multiply $f_s(t)$ by the unit step function to remove that part of $f_s(t)$ in the range $t < 0$. To eliminate the part of $f_s(t)$ for $t > t_0$, we can use the double-impulse weighting function $w(t)$ given by

$$w(t) = \delta(t) - \delta(t - t_0). \tag{31}$$

Convolving Eq. 31 with $f_s(t)$ is equivalent to adding to $f_s(t)$ its own negative delayed by one period. So performing these two operations in the *complex frequency domain*, by means of the Laplace transform we have

$$L\left[f_{(transient)}(t)\right]=\left(1-e^{-t_0 s}\right)\sum_{n=0}^{\infty}\frac{2\tilde{F}(n\Delta\omega)/t_0}{s-in\Delta\omega}, \tag{32}$$

which leads to the desired aperiodic time-domain representation

$$f_{(transient)}(t)=\sum_{n=0}^{\infty}\frac{2\tilde{F}(n\Delta\omega)}{t_0}\cdot\exp\{in\Delta\omega t\}\cdot\left[H(t)-H(t-t_0)\cdot\exp\{-in\Delta\omega t_0\}\right], \tag{33}$$

where $H(t)$ is the unit step function defined by Eq. 16. We observe that the first infinite sum of Eq. 32 is the Laplace transform of the semi-infinite periodic function $f_s(t)$ starting at $t = 0$ and the second infinite sum of Eq. 32 is the Laplace transform of an identical semi-infinite periodic function, $f_s(t-t_0)$, starting at $t=t_0$ that exactly cancels the first function from $t=t_0$ to $t=\infty$.

Also, $\tilde{F}(z,n\Delta\omega)=\tilde{E}_x(z,\omega)=\tilde{H}(z,\omega)\cdot\tilde{E}_x(0,\omega)$ from Eq. 3, where $\tilde{H}(z,\omega)$ is the medium's transfer function (not to be confused with *H(t)*, the Heaviside step function) and $\tilde{E}_x(z,\omega)$ is the Fourier transform of the propagating pulse at depth, *z*. If the time domain function $E_x(0,t)$ has duration T_1 and the "filter" impulse response $H(z,t)$, has a duration T_2, then the convolution result:

$$E_x(x,t)=H(z,t)\otimes E_x(0,t), \tag{34}$$

has a duration equal to T_1+T_2. If the "filtering" is done by multiplication in the frequency domain,

$$\tilde{F}(z,n\Delta\omega)=\tilde{E}_x(z,\omega)=\tilde{H}(z,\omega)\cdot\tilde{E}_x(0,\omega), \tag{35}$$

and $\Delta\omega$ is chosen as $2\pi/t_0$ with $t_0<(T_1+T_2)$, then overlapping in the time domain will occur and $E_x(z,t)$ will be distorted. If $T_1>T_2$, then $\Delta\omega=\dfrac{2\pi}{2T_1}$ is a safe choice and the overlapping effect (aliasing) in the time domain cannot occur.

Truncation Error

Next, we consider the error resulting from the evaluation of the integral over a finite frequency range, Ω. By truncating the integrand in Eq. 2 we obtain a *finite* inverse Fourier transform

$$\hat{f}(t)=\frac{1}{2\pi}\int_{-\Omega/2}^{\Omega/2}\tilde{F}(\omega)e^{i\omega t}d\omega. \tag{36}$$

Because the spectral densities of the pulsed waveforms under discussion have infinite tails, it is necessary to examine the relationship between the error in the computed inverse transform and the finite frequency interval Ω. By knowing this relationship it is possible to select the proper limits on the inverse transform to achieve a desired accuracy.

The finite inverse Fourier transform can be written as the infinite transform of a product of functions when one of those functions is the rectangular window function, $B(\omega)$ between $\Omega/2$ and $-\Omega/2$;

$$\hat{f}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{F}(\omega)\tilde{B}(\omega)e^{i\omega t}d\omega, \tag{37}$$

where

$$\tilde{B}(\omega) = \begin{cases} 1, & -\Omega/2 \le \omega \le \Omega/2 \\ 0, & \text{elsewhere} \end{cases}. \tag{38}$$

The inverse of the rectangular window function, $\tilde{B}(\omega)$, is obtained as follows

$$\begin{aligned} b(t) &= \frac{1}{2\pi}\int_{-\Omega/2}^{\Omega/2} 1\cdot e^{i\omega t}d\omega = \frac{1}{(2i)\pi t}\left\{e^{i\Omega/2t} - e^{-i\Omega/2t}\right\} \\ &= \frac{1}{\pi t}\sin\left(\frac{\Omega t}{2}\right) \\ &= \frac{\Omega}{2\pi}\operatorname{sinc}\left(\frac{\Omega t}{2}\right) \end{aligned} \tag{39}$$

Multiplication in the frequency domain is equivalent to convolution in the time domain. Therefore, we have for a pulse centered at the origin with duration equal to τ_0

$$f(t) = \int_{-\tau_0/2}^{+\tau_0/2} f(t-\tau)b(\tau)d\tau. \tag{40}$$

For example, if $f(t)$ were a single sinewave pulse of unity amplitude and carrier ω_c, we would have

$$f(t) = \frac{\Omega}{2\pi}\int_{-\tau_0/2}^{\tau_0/2} \sin[\omega_c(t-\tau)]\cdot[H(t-\tau+\tau_0/2) - H(t-\tau-\tau_0/2)]\cdot\operatorname{sinc}\left(\frac{\Omega\tau}{2}\right)d\tau. \tag{41}$$

Eq. 40 shows clearly how the finite inverse Fourier transform affects the computed time domain pulse. The distorted waveform is obtained by convolving the true value of the pulse waveform with the sinc function for each time point. The difference, $f(t) - f(t)$, is

the error due to the finite integration limits. The error obtained in this way is independent of the aliasing error due to sampling as discussed previously. Figures 16-18 show examples of this effect on a 1 radian/sec, four cycle, unit amplitude sinewave pulse. The distortion of the pulse caused by evaluating its inverse Fourier transform over progressively smaller bandwidths of Ω at 10, 5 and 2.5 radians/sec is demonstrated.

Results and Discussion

Sinewave Pulse in a Debye Medium

Our first set of results models the propagation of a single sinewave pulse with carrier frequency 1 GHz normally incident onto a water half-space (Debye medium). This incident pulse, shown in Figure 1, is a single burst of 10 carrier cycles. Figure 3 shows the transmitted pulse plotted against time at a depth of 0.75 meters in water. We note the attenuation with depth and the presence of the Brillouin precursors at the leading and trailing pulse edges. This result duplicates that obtained by previous investigators [1] who used a Fourier series calculation. The FORTRAN code listing for this case is included in Appendix-C.

Figure 4 illustrates the propagation of 2 1/2 pulses of a 5 pulse wavetrain in a water half-space showing how rapidly the carrier is attenuated. At 1 meter depth all that remains of the pulse is the Brillouin precursor pair associated with the leading and trailing edges of the pulse. Figure 5 shows the amplitude spectrum plotted over the positive frequencies for a finite duration pulse train consisting of five such pulses. Compared to the amplitude spectrum of a single pulse shown in Figure 2, Figure 5 reveals a more complex lobe structure due to the infinite sum of sinc functions needed to describe the spectrum of the five pulse train. Eq. A-19 and subsequent discussion in Appendix-A explain the origin of this more complex structure.

Sinewave Pulse in a Lorentz Medium

Our second set of results investigates the propagation of a finite sinewave pulse train (five pulses) normally incident onto a Lorentz half-space with a single resonance. As mentioned previously, the Lorentz model is appropriate for describing the absorption and transmission of infrared to optical signals in certain materials. Figure 6 shows two "snapshots" of a five-pulse burst taken at 60 and 100 femtoseconds after incidence. The carrier frequency of this pulse train lies near the upper end of the absorption band. The band-stop filtering action of the Lorentz medium blocks the pulse energy that falls within its absorption band. Only the energy outside the absorption band is transmitted. The spectral residue falling below the absorption band form a Brillouin precursor pair for each incident pulse. The spectral residue above the absorption band form a Sommerfeld precursor pair for each incident pulse. Since these two precursor types are separated in frequency by the width of the absorption band, one would expect them to move at different speeds in a linear dispersive medium such as this one. Comparing Figures 6 (a) & (b) indeed shows this to be true. We see the distance between the Sommerfeld and

Brillouin precursor pairs increasing with time which indicates a difference in propagation speeds. A series of plots similar to Figure 7 was generated to measure the difference in travel times for various distances up to 200 microns. A linear regression of displacement versus time for the Brillouin precursor pair in Figure 8 shows that this pair propagates at two-thirds the speed of light. Similarly, a linear regression was done for the Sommerfeld precursor pair of displacement versus time. Figure 9 indicates that the Sommerfeld precursor pair propagates at nearly vacuum light speed in this medium. Using the results for the Brillouin precursor speed and Snell's law yield a refraction angle for the Brillouin precursor of about 41° for light obliquely incident at 80° on a Lorentz half-space which agrees with the result reported in [4].

Gaussian Pulse in a Lorentz Medium

Our third set of results addresses the propagation of a wideband Gaussian transient into the same single resonance Lorentz medium as discussed above. This pulse is assumed to be normally incident on the Lorentz half-space, with pulse width parameter, τ, equal to .05 femtoseconds, and carrier frequency, $\omega_c = 5.75\text{E}+16$ rad/s. The goal was to match the results shown in Figure 8(a) of reference [3]. Our computation is based on the Fourier transform of a cosine carrier modulated with a Gaussian envelope as given in Appendix-B.

Figure 10 shows the spectrum of this pulse. We show in Figure 11 the time-history of this pulse at a depth of one micron in the Lorentz half-space. We note the close resemblance to Figure 8(a) in reference [3]. This result validates our Fourier integral numerical scheme by favorable comparison to another independent computational method based on the numerical inversion of the Laplace transform. The FORTRAN code for this case is included in Appendix-D.

Impulse Response of a Dielectric Medium

By setting $\tilde{E}_x(0,\omega)$ equal to unity in Eq. 10, we obtain the impulse response of the linear medium

$$h(z,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{H}(z,\omega)e^{i\omega t}d\omega. \quad \textbf{(42)}$$

The above integral can be computed by numerical quadrature using the same techniques previously discussed.

The time-domain convolution integral, provides an alternative means of obtaining the propagated time-domain pulse at depth z of the incident waveform $E_x(0,t)$ when the impulse response of the medium is available as follows

$$E_x(z,t) = \int_0^t E_x(0,t)h(z,t-\tau)d\tau. \quad \textbf{(43)}$$

In the evaluation of the above convolution integral by digital computer, it is necessary to replace the integral with a finite sum. We first replace the infinitesimal $d\tau$ with a time interval of finite duration T, and let $nT = \tau$. Then we can rewrite Eq. 43 as the summation

$$E_x(z,t) = T\sum_{n=0}^{K} E_x(0,nT)h(z,t-nT) \quad \textit{for } KT \le t < (K+1)T. \qquad \textbf{(44)}$$

Figure 12 shows the impulse response of pure water (Grant's model) at a depth of 0.75 meters where $h(z,t)$ is the water response measured at depth z for an impulse incident on the $z = 0$ plane. In Figure 13 we computed the time-history of a 10 ns pulse of 1 GHz at this depth by means of Eq. 44. We note the similarity to Figure 3 which was computed using the Fourier transform. The FORTRAN code for this case is included in Appendix-E.

Figure 14 shows the impulse response of the single absorption Lorentz medium at 1 micron depth. Once again using Eq. 44, we compute the time-history of this pulse in the Lorentz medium and see close agreement with Figure 7(a). The FORTRAN code for this case is given in Appendix-F.

Conclusion

We have shown how to compute the waveform of a transmitted pulse in a linear dispersive half-space by use of the pulse's Fourier transform. The approach discussed in this paper is applicable only to waveforms whose Fourier transform can be found analytically.[3] In particular, we have demonstrated this method for a finite train of sinewave pulses and a single Gaussian modulated sinewave transient. These two idealized waveforms reasonably approximate those often encountered in dosimetry computations involving "thick" homogenous layers (half-space) of bio-tissue and therefore should be useful in estimating electric field strengths in bio-dielectrics that can be accurately described by either the Debye or Lorentz models.

An unavoidable side-effect of uniformly sampling the spectral density function in order to numerically compute the inverse Fourier transform is that the resulting time-domain waveform is periodic with period equal to the reciprocal of the frequency sampling interval. We have shown how to analytically remove this periodicity and thereby recover the transient waveform.

[3] The Fourier transform method itself is not limited to analytically describable waveforms. It is possible to start with an arbitrary time waveform sampled at discrete time points and compute its Fourier transform using the DFT; multiply by the discretized medium transfer function and invert the result via the IDFT to obtain the propagated time-domain wave in the medium. The accuracy of this method is limited by the sampling resolution of the initial waveform. The desire to avoid this limitation was the motivation for choosing only analytically describable signals in this report.

Recommendations

Additional work might profitably be spent on refining these analytically convenient models of pulsed waveforms to make them more closely resemble real-world pulses. For example, the sinewave pulse should be modified to have a nonzero rise-time, as this is true for all band-limited pulses of physical origin. On the other hand, the Gaussian pulse should be modified to have truncated tails which is also true for all physical pulses of finite duration.

To increase the utility of these numerical pulse propagation models, it would be desirable to speed-up the computations by harnessing the speed of the Fast Fourier Transform (FFT) algorithm to evaluate the inverse Fourier transform. That is, the summation inherent in the evaluation of the Fourier integral by means of numerical quadrature would be performed by the inverse FFT algorithm.

References:

[1] R. Albanese, J.Penn and R. Medina, "Short-rise-time Microwave Pulse Propagation Through Dispersive Biological Media," J. Opt. Soc. Am. A, vol.6, no.9, pp 1441-1446 (1989)

[2] E.H. Grant, King's College, University of London, London, UK, personal communication to Dr. R.A. Albanese (1988)

[3] K.E. Oughstun, J.E.K. Laurens and C.M. Balictsis, "Asymptotic Description of Electromagnetic Pulse Propagation in a Linear Dispersive Medium," Ultra-Wideband, Short-Pulse Electromagnetics, pp 223-240, edited by H. Bertoni et al., Plenum Press (1993)

[4] J.G. Blaschak and J. Franzen, "Precursor Propagation in Dispersive Media from Short-rise-time Pulses at Oblique Incidence," J. Opt. Soc. Am. A,vol.12, no.7, pp 1501-1512 (1995)

[5] A. Sommerfeld, Optics, pp 88-101, Academic Press (1949)

[6] P. Debye, "Polar Molecules," Chemical Catalog, New York (1929)

[7] D. Eisenberg and W. Kauzmann, The Structure and Properties of Water, pp 206-214,Oxford University Press (1969)

[8] J. Roberts and F. Wang,"Dielectric Relaxation in Water Over the Frequency Range 13<f<18 GHz Using a Resonant Microwave Cavity Operating in the TM_{010} Mode," Journal of Microwave Power and Electromagnetic Energy, vol. 28, no. 4, pp 196-205 (1993)

[9] L. Brillouin, "About the Propagation of Light in Dispersive Media," Annalen der Physik, vol. 4, no.44 pp 203-240 (1914)

[10] L. Brillouin, Wave Propagation and Group Velocity, New York, Academic Press, (1960)

[11] K. Su, Time-Domain Synthesis of Linear Networks, pp 137-140 Prentice-Hall, (1971)

Appendix A Pulse train spectrum

All time domain pulsed signals can be described in terms of four time scales: wave period, pulsewidth, pulse repetition interval (PRI), and dwell time. Likewise, in the frequency-domain, pulsed signals can be described in terms of four corresponding frequency scales: carrier frequency, pulse bandwidth, pulse repetition frequency (PRF), and spectral line bandwidth [A1]. In Fourier analysis an inverse relationship exists between the time and frequency scales: the larger the wave period, the smaller the wave frequency.

This derivation of the Fourier transform of a *finite* number of sinewave pulses will illustrate the four frequency scales mentioned above.

For the finite regularly repeating signal shown in Fig. A1, $f_r(t)$ is the product of the infinite duration periodic function $f(t)$, and the gate function $g(t)$

$$f_r(t) = f(t) \cdot g(t) \tag{A-1}$$

where

$$\begin{aligned}
f(t) &= \sin(\omega_c t), \qquad -\tau/2 \le t \le \tau/2 \\
&= 0, \qquad \begin{cases} \tau/2 < t \le t_0 - \tau/2 \\ -t_0 + \tau/2 \le t < -\tau/2 \end{cases} \\
&= \sin(\omega_c t), \quad \begin{cases} t_0 - \tau/2 < t \le t_0 + \tau/2 \\ -t_0 - \tau/2 \le t < -t_0 + \tau/2 \end{cases} \\
&= 0, \qquad \begin{cases} t_0 + \tau/2 < t \le 2t_0 - \tau/2 \\ -2t_0 + \tau/2 \le t < -t_0 - \tau/2 \end{cases} \\
&\;\vdots \\
&\text{etc.,}
\end{aligned} \tag{A-2}$$

and

$$g(t) = \begin{cases} 1, & -T/2 \le t \le T/2 \\ 0, & \text{elsewhere} \end{cases} \tag{A-3}$$

Let an integer number, N, of carrier cycles occur during the pulse "on-time", τ, such that $\tau = 2\pi N / \omega_c$.

We begin by deriving the Fourier transform $F_0(\omega)$ of a single sinewave pulse, of finite duration, τ, unit amplitude, and symmetric with respect to the origin, given by the first line of Eq. A-2, which we call $f_0(t)$. We use the Fourier transform pair definition

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt \quad \text{and} \quad f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} d\omega \tag{A-4}$$

where $i = \sqrt{-1}$.

Since $f_0(t)$ is a finite duration sinewave and the sinewave is an odd function we see that

$$F[f_0(t)] = F_0(\omega) = -i \int_{-\infty}^{\infty} f_0(t) \sin(\omega t) dt \qquad \textbf{(A-5)}$$

and

$$F_0(\omega) = -i \int_{-\tau/2}^{\tau/2} \sin(\omega_c t) \sin(\omega t) dt \qquad \textbf{(A-6)}$$

Note that since $f_0(t)$ is real and odd, $F_0(\omega)$ will be imaginary and odd. Making use of a trigonometric identity we get

$$F_0(\omega) = -i \int_{-\tau/2}^{\tau/2} 1/2\{\cos[(\omega_c - \omega)t] - \cos[(\omega_c + \omega)t]\} dt \qquad \textbf{(A-7)}$$

Performing the integration, we have

$$F_0(\omega) = -i\left\{\left[\frac{\sin[(\omega_c - \omega)t]}{2(\omega_c - \omega)}\right]_{-\tau/2}^{\tau/2} - \left[\frac{\sin[(\omega_c + \omega)t]}{2(\omega_c + \omega)}\right]_{-\tau/2}^{\tau/2}\right\} \qquad \textbf{(A-8)}$$

$$F_0(\omega) = -i\left\{\frac{\sin[(\omega_c - \omega)\tau/2]}{(\omega_c - \omega)} - \frac{\sin[(\omega_c + \omega)\tau/2]}{(\omega_c + \omega)}\right\} \qquad \textbf{(A-9)}$$

$$F_0(\omega) = -i\left\{\frac{\sin[(\omega_c - \omega)N\pi/\omega_c]}{(\omega_c - \omega)} - \frac{\sin[(\omega_c + \omega)N\pi/\omega_c]}{(\omega_c + \omega)}\right\} \qquad \textbf{(A-10)}$$

Now it can be shown [A2], that the complex coefficients, c_n, of the Fourier series expansion of the periodic pulse train of sinusoidal pulses, $f(t)$, with repetition period, t_0, are equal to the values of the Fourier transform $F_0(\omega)$ of the single sinewave pulse, $f_1(t)$, evaluated at $\omega = n\omega_0 = n2\pi / t_0$ and multiplied by $1/t_0$, where $f_1(t)$ is defined by

$$f_1(t) = \begin{cases} f_0(t), & |t| \le t_0/2 \\ 0, & |t| > t_0/2 \end{cases} \qquad \textbf{(A-11)}$$

Therefore, the Fourier series coefficients of the infinite duration periodic pulsed sinewave train, $f(t)$, defined by Eq. A-2, are given by

$$c_n = \frac{1}{t_0} F_0(n\omega_0) = -\frac{i}{t_0}\left\{\frac{\sin[(\omega_c - n\omega_0)N\pi/\omega_c]}{(\omega_c - n\omega_0)} - \frac{\sin[(\omega_c + n\omega_0)N\pi/\omega_c]}{(\omega_c + n\omega_0)}\right\} \quad \textbf{(A-12)}$$

where

$$f(t) = \sum_{n=-\infty}^{n=+\infty} c_n e^{in\omega_0 t}; \qquad \omega_0 = 2\pi/t_0 \quad \textbf{(A-13)}$$

It can also be shown [A2] that the Fourier transform of any infinite train of pulses is given by

$$F(\omega) = 2\pi \sum_{n=-\infty}^{n=+\infty} c_n\, \delta(\omega - n\omega_0) \quad \textbf{(A-14)}$$

where c_n are the complex coefficients of the Fourier series expansion, and δ is the unit impulse function.

That is, the Fourier transform of a periodic function consists of a sequence of equidistant impulses located at the harmonic frequencies of the function with amplitudes equal to the complex coefficients of its Fourier series expansion. To obtain the Fourier transform of the *finite* pulse train of duration T, we need the Fourier transform of the gating function, $g(t)$, given by Eq. A-3. Its Fourier transform is the sinc function

$$F[g(t)] = G(\omega) = \frac{2}{\omega}\sin\left(\frac{\omega T}{2}\right) = T\,\mathrm{sinc}\left(\frac{\omega T}{2}\right) \quad \textbf{(A-15)}$$

To obtain a finite train of sinewave pulses, we multiply the periodic pulse train by the gating function. Next, we invoke the frequency convolution theorem to obtain the Fourier transform of a finite train of sinewave pulses

$$F[(f_r(t)] = F[(f(t)\cdot g(t)] = \frac{1}{2\pi} F(\omega) \otimes G(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\upsilon)G(\omega - \upsilon)d\upsilon \quad \textbf{(A-16)}$$

Convolving $F(\omega)$ with the transform of the gate function, $G(\omega)$, in the frequency domain yields

$$F[f_r(r)] = \frac{1}{2\pi} F(\omega) \otimes G(\omega) = \left\{ \sum_{n=-\infty}^{n=+\infty} c_n \, \delta(\omega - n\omega_0) \right\} \otimes G(\omega)$$

$$= \sum_{n=-\infty}^{n=+\infty} c_n \, \delta(\omega - n\omega_0) \otimes G(\omega)$$

$$= \sum_{n=-\infty}^{n=+\infty} c_n \, G(\omega - n\omega_0) \qquad \textbf{(A-17)}$$

$$= \sum_{n=-\infty}^{n=+\infty} c_{-n} \, G(\omega + n\omega_0)$$

$$= \sum_{n=-\infty}^{n=+\infty} c^*{}_n \, G(\omega + n\omega_0)$$

Substituting Eq. A-15 into A-17, we finally obtain

$$F[f_r(t)] = T \left\{ \sum_{n=-\infty}^{n=+\infty} c_n^* \frac{\sin[(\omega + n\omega_0)T/2]}{(\omega + n\omega_0)T/2} \right\} \qquad \textbf{(A-18)}$$

$$F[f_r(t)] = \frac{2i}{t_0} \sum_{n=-\infty}^{n=+\infty} \frac{\sin[(\omega_c - n\omega_0)N\pi/\omega_c]\sin[(\omega + n\omega_0)T/2]}{(\omega_c - n\omega_0)(\omega + n\omega_0)}$$
$$- \frac{2i}{t_0} \sum_{n=-\infty}^{n=+\infty} \frac{\sin[(\omega_c + n\omega_0)N\pi/\omega_c]\sin[(\omega + n\omega_0)T/2]}{(\omega_c + n\omega_0)(\omega + n\omega_0)}$$

$$\textbf{(A-19)}$$

$$= \frac{2i}{t_0} \sum_{n=-\infty}^{n=+\infty} \frac{\sin[(\omega_c - n\omega_0)\tau/2]\sin[(\omega + n\omega_0)T/2]}{(\omega_c - n\omega_0)(\omega + n\omega_0)}$$
$$- \frac{2i}{t_0} \sum_{n=-\infty}^{n=+\infty} \frac{\sin[(\omega_c + n\omega_0)\tau/2]\sin[(\omega + n\omega_0)T/2]}{(\omega_c + n\omega_0)(\omega + n\omega_0)}$$

The Fourier transform of the periodic sinewave pulse train of finite duration T (as are all physical signals) is the sum of terms given by Eq. A-19, where ω_0 is the repetition rate of the pulses in the train. Each term of this sum can be plotted as an individual amplitude distribution centered at $\omega = n\omega_0$. As the duration T increases, the amplitude distributions become more compact in the frequency dimension. As T goes to infinity, the total amplitude distribution approaches a line spectrum.

Thus, four frequency scales are required to completely describe a finite train of pulsed sinewaves: the bandwidth of the spectral lines ($1/T$); spacing of the spectral lines ($\mathrm{PRF} = 1/t_0$); bandwidth of the sinc function envelope ($1/\tau$); and the carrier frequency f_c.

References:

[A1] Morris, G.V., Airborne Pulsed Doppler Radar, pp 35-40, Artech House, (1988)

[A2] Papoulis, A., The Fourier Integral and Its Applications, pp 43-45, McGraw-Hill, (1962)

Appendix B Gaussian spectrum

In this appendix we derive the Fourier transform of an amplitude modulated cosine carrier where the modulation waveform is the Gaussian envelope. Let a Gaussian pulsed sinusoid of frequency ω_0, amplitude A and pulse width $\sqrt{2}\tau$ be represented by the real part of

$$f(t) = Ae^{-t^2/(2\tau)^2} e^{i\omega_0 t} \qquad \textbf{(B-1)}$$

where $\sqrt{2}\tau =$ standard deviation . Since this function is symmetric with respect to the origin (even), its spectral function F(ω) is also even. We calculate its spectral density function as follows

$$f(t) = A \cdot \mathrm{Re}\left\{ \exp\left[\frac{-t^2}{(2\tau)^2} + i\omega_0 t \right] \right\} = Ae^{\frac{-t^2}{(2\tau)^2}} \left\{ \frac{e^{i\omega_0 t} + e^{-i\omega_0 t}}{2} \right\} \qquad \textbf{(B-2)}$$

Taking the Fourier transform of Eq. B-2, we have

$$F(\omega) = \frac{A}{2} \int_{-\infty}^{\infty} \exp\left[-i\omega t + i\omega_0 t - \frac{t^2}{4\tau^2} \right] dt + \frac{A}{2} \int_{-\infty}^{\infty} \exp\left[-i\omega t - i\omega_0 t - \frac{t^2}{4\tau^2} \right] dt \qquad \textbf{(B-3)}$$

Completing the square by adding and subtracting $[i\tau(\omega - \omega_0)]^2$ inside the exponent of the first integral, and similarly adding and subtracting $[i\tau(\omega + \omega_0)]^2$ inside the exponent of the second integral, we obtain

$$\begin{aligned} F(\omega) = {} & \frac{A}{2} \int_{-\infty}^{\infty} \exp\left[-\left(\frac{t}{2\tau} + i\tau(\omega - \omega_0) \right)^2 - \tau^2 (\omega - \omega_0)^2 \right] dt \\ & + \frac{A}{2} \int_{-\infty}^{\infty} \exp\left[-\left(\frac{t}{2\tau} + i\tau(\omega + \omega_0) \right)^2 - \tau^2 (\omega + \omega_0)^2 \right] dt \end{aligned} \qquad \textbf{(B-4)}$$

Next, we make a change of variable in each integral. Designate the first squared term in the exponential as *x*, then $dt = 2\tau dx$. After making this substitution the limits of integration remain infinite, and we have

$$F(\omega) = A\tau e^{-\tau^2(\omega-\omega_0)^2} \int_{-\infty}^{\infty} e^{-x^2} dx + A\tau e^{-\tau^2(\omega+\omega_0)^2} \int_{-\infty}^{\infty} e^{-x^2} dx \qquad \textbf{(B-5)}$$

From the integral tables we find that

$$\sqrt{\pi} = \int_{-\infty}^{\infty} e^{-x^2} dx \qquad \textbf{(B-6)}$$

So Eq. B-5 simplifies to

$$F(\omega) = \sqrt{\pi} A \tau e^{-\tau^2(\omega-\omega_0)^2} + \sqrt{\pi} A \tau e^{-\tau^2(\omega+\omega_0)^2} \qquad \textbf{(B-7)}$$

which is the sum of two Gaussian curves shifted up and down by the angular frequency of the carrier. If the original pulse, Eq. B-1, is narrowed by making τ smaller, the spectral density curve, Eq. B-7, becomes broader with reduced amplitude exhibiting the inverse relationship between temporal width and spectral width in accordance with the scaling property of the Fourier transform:

$$f(t/\tau) \Longleftrightarrow |\tau| F(\tau\omega) \qquad \textbf{(B-8)}$$

Finally, we note that the Fourier transform of the Gaussian modulated sinewave is real-valued.

Appendix C FORTRAN codes sinewave-Debye case

```
        program SDT
**********************************************************************
* SDT: Propagates a finite train of sinewave pulses into a Debye
* half-space using the Fourier transform and the Fresnel trans-
* mission coefficient for normal incidence (air/water interface).
*
* ... Z FIXED, TIME VARIES...
*
*              by J.Franzen 7-12-95
**********************************************************************
        parameter( npts = 4000, mpts = 2000 )
        complex*16 j, FOUR(npts), ZERO, k, Et, eps, DENOM, ior, E
        implicit real*8 (a-h,o-z)
        logical ex

        pi = dacos(-1d0)
        twopi = 2d0*pi
        ZERO = dcmplx(0d0,0d0)
        j = dcmplx(0d0,1d0)

        inquire (file='SD_spect.dat',exist=ex)
        if (ex) then
         open(9,file='SD_spect.dat',status='OLD')
         rewind 9
        else
         open(9,file='SD_spect.dat',status='NEW')
        end if

        fc = 1d0                ! 1 GHz Carrier
        fmax = 1d1              ! (+/-) 10 GHz Integration Limit
        tau = 1d1               ! 10 nsec On-time = 10 Cycles
        prf = 1d0/2d1           ! PRF = 1/PRI = 1/20 = 50% Duty Cycle
        dwell = 1d2             ! FIVE Pulses
*     dwell = 1d1             ! ONE Pulse
        df = fmax/dfloat(npts)
        print*,'Repetition period = ',1d0/df,' nanosec'
        dw = twopi*df

Check if Fourier spectrum already exists. If so, read into array FOUR(n).
        if (ex) then
         print*,'Reading Fourier spectrum...'
         do n=1,npts
          read (9,200) f, FOUR(n)
         end do
         print*,'Done!'
         goto 1
        end if

        print*,'Begin ',npts,' point Fourier transform now....'

        call fourier(npts, fmax, fc, tau, prf, dwell,   FOUR, dw)
```

```
        print*,'End of Fourier transform computation!'

        do n=1,npts
         f = (n-1)*df
         write(9,200) f, dreal(FOUR(n)), dimag(FOUR(n))
200      format(1x,f10.5,2x1pE20.12,2x1pE20.12)
        end do
      close (unit= 9)

1     CONTINUE

      write(*,'(" Please enter z (m): ",$)')
      read*,z
      write(*,'(" Plz enter t-min (ns): ",$)')
      read*,tmin
      write(*,'(" Plz enter t-max (ns): ",$)')
      read*,tmax

      inquire (file='SDT.dat',exist=ex)
      if (ex) then
       open(10,file='SDT.dat',status='OLD')
       rewind 10
      else
       open(10,file='SDT.dat',status='NEW')
      end if

      c = .2997d0    ! light speed (m/ns)
      tau = 8.1d-3   ! relaxation const. scaled to 19.648758 GHz loss peak
      eps_0 = 1d7/(4d0*pi*c**2)   ! Permittivity of free space
      it = 0
      dt = (tmax-tmin)/dfloat(mpts)
      do t = tmin, tmax, dt
       Et = ZERO
       do n = 1,npts
        w = dfloat(n-1)*dw
        DENOM = dcmplx(1d0, w*tau)
        eps = 5.5d0 + 72.7d0/DENOM
           if (w .ne. 0d0) then
         eps = eps - j*(1d-5)/w/eps_0    ! ... subtract ionic conductivity
           end if
        ior = cdsqrt(eps)
*       k = w/c*ior
           k = w/c                            ! ... no dispersion, for check out
           E = FOUR(n)*cdexp(-j*(k*z - w*t)) ! ... no dispersion, for check out
*          E = FOUR(n)*2d0/(1d0+ior)*cdexp(-j*(k*z - w*t))
        Et = Et + E

* Trapezoidal rule end-point correction:
        if(n.eq.1 .or. n.eq.npts) then
            Et = Et - .5d0*E
        end if
       end do
```

```
      it = it + 1
      write(10,200) t,dreal(Et)/twopi*dw
*     write(* , * ) t,dreal(Et)/twopi*dw
      if(mod(it,100).eq.0) then
        write(*,*) ' it = ',it
      end if
     end do

     close (unit=10)
     stop
     E N D
     subroutine fourier( npts, fmax, fc, tau, prf, dwell,   four, dw )
****************************************************************************
*  Computes the Fourier transform of a FINITE train of pulsed sine-
*  waves.
*  Modified to compute spectrum for positive frequencies only.
*
*  Inputs:
*    npts - Number of points at which to compute the Fourier spectrum
*    fmax - Maximum frequency range of Fourier spectrum (Hz)
*      fc - Carrier frequency of pulse train (Hz)
*     tau - Pulse on-time (sec)
*     prf - Pulse repetition frequency (Hz)
*   dwell - Duration of pulse train (sec)
*
* Outputs:
*    four - Complex array of size npts containing Fourier spectrum
*      dw - Frequency resolution: fmax/npts*twopi (rad/sec)
*
*                by J.Franzen 5-24-95
****************************************************************************
     IMPLICIT REAL*8 (a-h,o-z)
     COMPLEX*16 j, ZERO, FOUR(npts)

     sinc(q) = dsin(q)/q
     pi = dacos(-1d0)
     twopi = 2d0*pi
     j = dcmplx(0d0,1d0)
     ZERO = dcmplx(0d0,0d0)
     do n = 1, npts
      FOUR(n) = ZERO
     end do

     omega_0 = 2d0*pi*prf
     omega_c = 2d0*pi*fc
     t_0 = 1d0/prf
     T = dwell

     dw = twopi*fmax / dfloat(npts)
     do n = 1, npts
         w = dfloat(n-1)*dw

         do L = -1000, 1000, 1
           sum = w + L*omega_0
```

```
      sum = sum * T/2d0
      dif = omega_c - L*omega_0
      dif = dif * tau/2d0
      if (sum .eq. 0d0) then
        ap = tau*T/4d0*sinc(dif)
      else if (dif .eq. 0d0) then
        ap = tau*T/4d0*sinc(sum)
      else if (sum .eq. 0d0 .and. dif .eq. 0d0) then
        ap = tau*T/4d0
      else
        ap = tau*T/4d0*sinc(dif)*sinc(sum)
      end if

      sum1 = omega_c + L*omega_0
      sum1 = sum1 * tau/2d0
      if (sum .eq. 0d0) then
        an = tau*T/4d0*sinc(sum1)
      else if (sum1 .eq. 0d0) then
        an = tau*T/4d0*sinc(sum)
      else if (sum .eq. 0d0 .and. sum1 .eq. 0d0) then
        an = tau*T/4d0
      else
        an = tau*T/4d0*sinc(sum1)*sinc(sum)
      end if

      FOUR(n) = FOUR(n) + ap - an
     end do
     FOUR(n) = 4d0*j/t_0*FOUR(n)
     if (mod(n,100).eq.0) then
      print*,'n = ',n
     end if
end do

RETURN
E N D
```

```
        program SDT2
*********************************************************************
* SDT2: Propagates a finite train of sinewave pulses into a Debye
* half-space using the Fourier transform and the Fresnel trans-
* mission coefficient for normal incidence (air/water interface).
*
* ... Z FIXED, TIME VARIES...
*
* ... MODIFIED to remove peroidicity
*
*                by J.Franzen 2-05-96
*********************************************************************
        parameter( npts = 4000, mpts = 2000 )
        complex*16 j, FOUR(npts), ZERO, k, Et, eps, DENOM, ior, E
        implicit real*8 (a-h,o-z)
        logical ex

        pi = dacos(-1d0)
        twopi = 2d0*pi
        ZERO = dcmplx(0d0,0d0)
        j = dcmplx(0d0,1d0)

        inquire (file='SD_spect.dat',exist=ex)
        if (ex) then
         open(9,file='SD_spect.dat',status='OLD')
         rewind 9
        else
         open(9,file='SD_spect.dat',status='NEW')
        end if

        fc = 1d0                  ! 1 GHz Carrier
        fmax = 1d1                ! (+/-) 10 GHz Integration Limit
        tau = 1d1                 ! 10 nsec On-time = 10 Cycles
        prf = 1d0/2d1             ! PRF = 1/PRI = 1/20 = 50% Duty Cycle
        dwell = 1d2               ! FIVE Pulses
*       dwell = 1d1               ! ONE Pulse
        df = fmax/dfloat(npts)
      period = 1d0/df
        print*,'Repetition period = ',period,' nanosec'
        dw = twopi*df

Check if Fourier spectrum already exists. If so, read into array FOUR(n).
        if (ex) then
         print*,'Reading Fourier spectrum...'
         do n=1,npts
          read (9,200) f, FOUR(n)
         end do
         print*,'Done!'
         goto 1
        end if

        print*,'Begin ',npts,' point Fourier transform now....'

        call fourier(npts, fmax, fc, tau, prf, dwell,   FOUR, dw)
```

```
      print*,'End of Fourier transform computation!'

      do n=1,npts
       f = (n-1)*df
       write(9,200) f, dreal(FOUR(n)), dimag(FOUR(n))
200    format(1x,f10.5,2x1pE20.12,2x1pE20.12)
      end do
    close (unit= 9)

1     CONTINUE

    write(*,'(" Please enter z (m): ",$)')
    read*,z
    write(*,'(" Plz enter t-min (ns): ",$)')
    read*,tmin
    write(*,'(" Plz enter t-max (ns): ",$)')
    read*,tmax

    inquire (file='SDT.dat',exist=ex)
    if (ex) then
     open(10,file='SDT.dat',status='OLD')
     rewind 10
    else
     open(10,file='SDT.dat',status='NEW')
    end if

    c = .2997d0    ! light speed (m/ns)
    tau = 8.1d-3   ! relaxation const. scaled to 19.648758 GHz loss peak
    eps_0 = 1d7/(4d0*pi*c**2)  ! Permittivity of free space
    it = 0
    dt = (tmax-tmin)/dfloat(mpts)
    do t = tmin, tmax, dt
     Et = ZERO
     do n = 1,npts
      w = dfloat(n-1)*dw
      DENOM = dcmplx(1d0, w*tau)
      eps = 5.5d0 + 72.7d0/DENOM
        if (w .ne. 0d0) then
       eps = eps - j*(1d-5)/w/eps_0   ! ... subtract ionic conductivity
        end if
      ior = cdsqrt(eps)
      k = w/c*ior
*       k = w/c                          ! ... no dispersion, for check out
*       E = FOUR(n)*cdexp(-j*(k*z - w*t)) ! ... no dispersion, for check out
*       E = FOUR(n)*2d0/(1d0+ior)*cdexp(-j*(k*z - w*t))

     E = U(t)*FOUR(n)*2d0/(1d0+ior)*cdexp(-j*(k*z - w*t)) -
    &U(t-period+dwell/2.)*FOUR(n)*2d0/(1d0+ior)*
    &cdexp(-j*(k*z-w*(t-period)))

     Et = Et + E

* Trapezoidal rule end-point correction:
```

```
     if(n.eq.1 .or. n.eq.npts) then
          Et = Et - .5d0*E
     end if
    end do

    it = it + 1
    write(10,200) t,dreal(Et)/twopi*dw
*    write(* , * ) t,dreal(Et)/twopi*dw
    if(mod(it,100).eq.0) then
      write(*,*) ' it = ',it
    end if
   end do

   close (unit=10)
   stop
   E N D
   subroutine fourier( npts, fmax, fc, tau, prf, dwell,   four, dw )
****************************************************************************
*  Computes the Fourier transform of a FINITE train of pulsed sine-
*  waves.
*  Modified to compute spectrum for positive frequencies only.
*
*  Inputs:
*    npts - Number of points at which to compute the Fourier spectrum
*    fmax - Maximum frequency range of Fourier spectrum (Hz)
*      fc - Carrier frequency of pulse train (Hz)
*     tau - Pulse on-time (sec)
*     prf - Pulse repetition frequency (Hz)
*   dwell - Duration of pulse train (sec)
*
* Outputs:
*    four - Complex array of size npts containing Fourier spectrum
*      dw - Frequency resolution: fmax/npts*twopi (rad/sec)
*
*                   by J.Franzen 5-24-95
****************************************************************************
   IMPLICIT REAL*8 (a-h,o-z)
   COMPLEX*16 j, ZERO, FOUR(npts)

   sinc(q) = dsin(q)/q
   pi = dacos(-1d0)
   twopi = 2d0*pi
   j = dcmplx(0d0,1d0)
   ZERO = dcmplx(0d0,0d0)
   do n = 1, npts
    FOUR(n) = ZERO
   end do

   omega_0 = 2d0*pi*prf
   omega_c = 2d0*pi*fc
   t_0 = 1d0/prf
   T = dwell

   dw = twopi*fmax / dfloat(npts)
```

```
do n = 1, npts
     w = dfloat(n-1)*dw

     do L = -1000, 1000, 1
       sum = w + L*omega_0
       sum = sum * T/2d0
       dif = omega_c - L*omega_0
       dif = dif * tau/2d0
       if (sum .eq. 0d0) then
         ap = tau*T/4d0*sinc(dif)
       else if (dif .eq. 0d0) then
         ap = tau*T/4d0*sinc(sum)
       else if (sum .eq. 0d0 .and. dif .eq. 0d0) then
         ap = tau*T/4d0
       else
         ap = tau*T/4d0*sinc(dif)*sinc(sum)
       end if

       sum1 = omega_c + L*omega_0
       sum1 = sum1 * tau/2d0
       if (sum .eq. 0d0) then
         an = tau*T/4d0*sinc(sum1)
       else if (sum1 .eq. 0d0) then
         an = tau*T/4d0*sinc(sum)
       else if (sum .eq. 0d0 .and. sum1 .eq. 0d0) then
         an = tau*T/4d0
       else
         an = tau*T/4d0*sinc(sum1)*sinc(sum)
       end if

       FOUR(n) = FOUR(n) + ap - an
     end do
     FOUR(n) = 4d0*j/t_0*FOUR(n)
     if (mod(n,100).eq.0) then
       print*,'n = ',n
     end if
end do

RETURN
E N D
     real*8 function U(t)
     if (t.ge.0d0) then
       U = 1d0
     else
       U = 0d0
     end if
     return
     end
```

Appendix D FORTRAN code Gaussian-Lorentz case.

```
        program GLT2
*************************************************************************
* GLT2: Propagates a single Gaussian pulse into a Lorentzian half-
* space using the Fourier transform and the Fresnel transmission
* coefficient for normal incidence.
*
*************************************************************************
**** Modified to use Oughstun's nondimensional space-time parameter
**** THETA
*************************************************************************
*
* ... Z FIXED, TIME VARIES ...
*
*              by J.Franzen 9-29-95
*************************************************************************
        parameter( npts = 2**13, mpts = 10 000 )
        implicit real*8 (a-h,o-z)
        complex*16 j, FOUR(npts), ZERO, Et,E, eps, DENOM
        complex*16 T12(npts),k(npts),ior(npts)
        logical ex

        pi = dacos(-1d0)
        twopi = 2d0*pi
        ZERO = dcmplx(0d0,0d0)
        j = dcmplx(0d0,1d0)
        c = 2.997d8 / 1d16        ! light speed (m/deci-femto-sec)

* Set values for demonstration:
        fc = .915d0          ! Carrier Freq;  X 10**16 Hz
        fmax = 1d1           ! Integration Limit;  X 10**16 Hz
        tau = .25d0     ! On-time;  X 10**(-16) sec
        df = fmax/dfloat(npts)
     print*,'Repetition period = ',1d-1/df,' femtosec'
        dw = twopi*df

Check if Fourier spectrum already exists. If so, read into array FOUR(n).
        inquire (file='GLT2_spect.dat',exist=ex)
        if (ex) then
         open(9,file='GLT2_spect.dat',status='OLD')
         rewind 9
        else
         open(9,file='GLT2_spect.dat',status='NEW')
        end if

        if (ex) then
         print*,'Reading Fourier spectrum...'
         do n=1,npts
          read (9,200) f, FOUR(n)
         end do
         print*,'Done!'
         goto 1
        end if
```

```
        print*,'Begin ',npts,' point Fourier transform now....'

        call gfour(npts, fmax, fc, tau,   FOUR, dw)

        print*,'End of Fourier transform computation!'

        do n=1,npts
         f =  (n-1)*df
         write(9,200) f, FOUR(n)
200     format(1x,f10.5,2x1pE20.12,2x1pE20.12)
        end do
        close (unit= 9)

1     CONTINUE

Compute Refl., Trans. coefs & k-numbers for LORENTZ half-space.
    omega_0 = 4d0        ! Resonant Frequency; X 10**16 rad/s
    delta = .28d0        ! Damping Constant;   X 10**16 rad/s
    do n=1,npts
        w = dfloat(n-1)*dw
        DENOM = w**2 - 2d0*j*w*delta - omega_0**2
        eps = 1d0 - (1.25d0*omega_0**2)/DENOM
        ior(n) = cdsqrt(eps)
        T12(n) = 2d0/( 1d0 + ior(n) )      ! TE Polarization
        k(n) = w/c*ior(n)
    end do

    write(*,'(" Please enter z (m): ",$)')
    read*,z
    print*,'z (microns) = ',z*1d6
    write(*,'(" Plz enter t-min (fs): ",$)')
    read*,tmin
    print*,'t-min = ',tmin
    tmin = tmin*1d1
    write(*,'(" Plz enter t-max (fs): ",$)')
    read*,tmax
    print*,'t-max (fs) = ',tmax
    tmax = tmax*1d1

    inquire (file='GLT2.dat',exist=ex)
    if (ex) then
        open(10,file='GLT2.dat',status='OLD')
        rewind 10
    else
        open(10,file='GLT2.dat',status='NEW')
    end if

* For each value of z, compute the inverse Fourier integral by
* numerical quadrature using the trapezoidal rule.
    it = 0
    dt = (tmax-tmin)/dfloat(mpts)
    do t = tmin, tmax, dt
     theta = c*t/z
```

```
      it = it + 1
      Et = ZERO
      do n = 1,npts
         w = dfloat(n-1)*dw
*        E = FOUR(n)*T12(n)*cdexp(-j*(k(n)*z - w*t) )
         E = FOUR(n)*T12(n)*cdexp(-j*z/c*w*(ior(n)-theta))
         Et = Et + E        ! Transmitted Field only

* Trapezoidal rule end-point corrections:
        if(n.eq.1 .or. n.eq.npts) then
          Et = Et - .5d0*E
         end if

      end do      ! End of w-loop....

      Et = Et/twopi*dw

      if(mod(it,100).eq.0) then
         write(*,*) ' it = ',it
      end if
      write(10,201) theta, -dreal(Et)
201   format(1x,2x1pE20.12,2x1pE20.12)

      end do      ! End of t-loop....

      close (unit=10)
      stop
      E N D
      subroutine gfour( npts, fmax, fc, tau,   four, dw )
*************************************************************************
*  Computes the Fourier transform of a Gaussian pulse.
*
*  Inputs:
*    npts - Number of points at which to compute Fourier spectrum
*    fmax - Maximum frequency range of Fourier spectrum (Hz)
*     tau - sigma/sqrt(2)
*
* Outputs:
*    four - Complex array of size npts containing Fourier spectrum
*      dw - Frequency resolution: fmax/npts*twopi (rad/sec)
*
*                 by J.Franzen 7-06-95
*************************************************************************

      IMPLICIT REAL*8 (a-h,o-z)
      COMPLEX*16 j, ZERO, FOUR(npts)

      pi = dacos(-1d0)
      twopi = 2d0*pi
      j = dcmplx(0d0,1d0)
      ZERO = dcmplx(0d0,0d0)

      omega_c = 2d0*pi*fc
      dw = twopi*fmax / dfloat(npts)
      do n = 1, npts
```

```
        w = dfloat(n-1)*dw
        FOUR(n) = dexp(-(tau**2)*(w-omega_c)**2)
        FOUR(n) = j*dsqrt(2d0*twopi)*tau*FOUR(n)
*       print*,'four(',n,') = ',four(n)
        if (mod(n,100).eq.0) then
         print*,'n = ',n
        end if
    end do

    RETURN
    END
```

Appendix E FORTRAN code water impulse response

```
      program IMP22
**********************************************************************
* IMP22: Performs time-domain convolution on sinewave pulse using
*     the impulse response of water.
*
*
*                   by J.Franzen 2-01-96
**********************************************************************
   parameter( npts = 4000, mpts = 8000 )
   complex*16 j, ZERO, k, eps, DENOM, ior, E
   implicit real*8 (a-h,o-z)
   real*8 IMP(mpts),tau(mpts),X(npts),Y(npts+mpts-1)
   logical ex

   pi = dacos(-1d0)
   twopi = 2d0*pi
   ZERO = dcmplx(0d0,0d0)
   j = dcmplx(0d0,1d0)

   fmax = 2d9          ! (+/-) 20 GHz Integration Limit
   df = fmax/dfloat(npts)
   dw = twopi*df

   write(*,'(" Please enter z (m): ",$)')
   read*,z

Check if Impulse response already exists. If so, read into array IMP(m).
   inquire (file='IMP22.dat',exist=ex)
   if (ex) then
    open(10,file='IMP22.dat',status='OLD')
    rewind 10
   else
    open(10,file='IMP22.dat',status='NEW')
   end if

   if (ex) then
       print*,'Reading impulse response...'
       do m=1,mpts
        read (10,200) tau(m),IMP(m)
       end do
       print*,'Done!'
       goto 1
   end if

Compute Impulse Response ...
   c = .2997d9       ! Light speed (m/s)
   taw = 8.1d-12     ! Water relaxation const.
   eps_0 = 8.854d-12 ! Permittivity of free space
   it = 0
```

```
      dT = 1d0/(2d0*fmax)
      do m = 1,mpts
       IMP(m) = 0d0
       do n = 1,npts
        w = dfloat(n-1)*dw
        DENOM = dcmplx(1d0, w*taw)
        eps = 5.5d0 + 72.7d0/DENOM
           if (w .ne. 0d0) then
          eps = eps - j*(1d-5)/w/eps_0   ! ... subtract ionic conductivity
           end if
        ior = cdsqrt(eps)
        k = w/c*ior
*          k = w/c                        ! ... no dispersion, for check out
*          E = cdexp(-j*(k*z - m*w*dT))   ! ... no dispersion, for check out
           E = 2d0/(1d0+ior)*cdexp(-j*(k*z - m*w*dT))
*       IMP(m)=IMP(m)+dreal(E)*dcos(m*pi*dT)-dimag(E)*dsin(m*pi*dT)
           IMP(m) = IMP(m) + dreal(E)
       end do

       it = it + 1
       tau(m) = m*dT
       IMP(m) = 2d0*IMP(m)/twopi*dw
       write(10,200) tau(m),IMP(m)
200    format(1x,1pe14.6,2x,1pe14.6)
       if(mod(it,100).eq.0) then
        write(*,*) ' it = ',it
       end if
      end do
      close (unit=10)

1     CONTINUE

Compute the Convolution Integral:
           inquire (file='PULSE22.dat',exist=ex)
           if (ex) then
            open(11,file='PULSE22.dat',status='OLD')
            rewind 11
           else
            open(11,file='PULSE22.dat',status='NEW')
           end if

           fc = 1d9          !!!  1 GHz Carrier Frequency
           on_time = 1d-8 !!!  10 ns on-time (10 cycles)
           it = 0
        dT = 1d0/(2d0*fmax)
           do n=1,npts
            t = n*dT
            X(n) = pulse(t, fc,on_time)
           end do

           call CONVL (X, npts, IMP, mpts, Y, ly)

           do n=1,ly
            write(11,200) n*dT, Y(n)*dT
```

```
      end do
      close (unit=11)

      stop
      E N D

      real*8 function pulse (t, fc,on_time)
      implicit real*8 (a-h,o-z)
      pi = dacos(-1d0)

      if (t .lt. on_time) then
        pulse = dsin(2d0*pi*fc*t)
      else
        pulse = 0d0
      end if

      return
      END

      subroutine CONVL (X, n, H, m, Y, ly)
*
*     Subroutine CONVL computes the convolution
*     between the sequences X and H
*     Parameters
*       X    : array containing sequence X
*       n    : length of sequence X
*       H    : array containing sequence H
*       m    : length of sequence H
*       Y    : array containing the convolution of X and H
*       ly   : length of the sequence Y
*
      real*8 X,H,Y
      dimension X(1),H(1),Y(1)

      ly = n + m - 1
      do  k = 1 , ly
        Y(k) = 0d0
      end do

      do 1 i = 1 , n
        do 1 j = 1 , m
          k = i + j - 1
1         Y(k) = Y(k) + X(i) * H(j)

      return
      END
```

Appendix F FORTRAN code Lorentz medium impulse response

```
        program IMP3
************************************************************************
* IMP3: Performs time-domain convolution on sinewave pulse using the
*     impulse response of a single absorption band Lorentz modim.
*
*                 1 Pulse @ 8d15 Hz carrier, 10 cycles-on
*
*
*             by J.Franzen 10-23-95
************************************************************************
        parameter( npts = 2**13, mpts = 2**14 )
        implicit real*8 (a-h,o-z)
        complex*16 j, ZERO, E, eps, DENOM
        complex*16 T12(npts),k(npts),ior
        real*8 IMP(mpts),tau(mpts),X(npts),Y(npts)
        logical ex

        pi = dacos(-1d0)
        twopi = 2d0*pi
        ZERO = dcmplx(0d0,0d0)
        j = dcmplx(0d0,1d0)
        c = 2.997d8 / 1d16       ! light speed (m/deci-femto-sec)

* Set values for demonstration:
        fc = .8d0             ! Carrier Freq;  X 10**16 Hz
        fmax = 2d1            ! Integration Limit;  X 10**16 Hz
        taw = 12.5d0          ! On-time;  X 10**(-16) sec
        df = fmax/dfloat(npts)
     print*,'Repetition period = ',1d-1/df,' femtosec'
        dw = twopi*df

Check if impulse response already exists. If so, read into array IMP(m).
        inquire (file='IMP3.dat',exist=ex)
        if (ex) then
         open(9,file='IMP3.dat',status='OLD')
         rewind 9
        else
         open(9,file='IMP3.dat',status='NEW')
        end if

        if (ex) then
         print*,'Reading impulse rsponse ...'
         do n=1,mpts
          read (9,200) tau(m),IMP(m)
         end do
         print*,'Done!'
         goto 1
        end if
```

```
Compute impulse response ...
    omega_0 = 4d0        ! Resonant Frequency; X 10**16 rad/s
    delta = .28d0        ! Damping Constant;   X 10**16 rad/s
    do n=1,npts
        w = dfloat(n-1)*dw
        DENOM = w**2 - 2d0*j*w*delta - omega_0**2
        eps = 1d0 - (1.25d0*omega_0**2)/DENOM
        ior = cdsqrt(eps)
        T12(n) = 2d0/( 1d0 + ior )
        k(n) = w/c*ior
    end do

    write(*,'(" Please enter z (microns): ",$)')
    read*,z
    print*,'z (microns) = ',z
    z = z*1d-6

    it = 0
    dT = 1d0/(2d0*fmax)
    do m = 1, mpts
     it = it + 1
     IMP(m) = 0d0
     tau(m) = m*dT
     do n = 1,npts
         w = dfloat(n-1)*dw
         E = T12(n)*cdexp(-j*(k(n)*z - w*m*dT) )
         IMP(m) = IMP(m) + dreal(E)
     end do

     IMP(m) = 2d0*IMP(m)/twopi*dw
     write(9,200) tau(m), IMP(m)
     if(mod(it,100).eq.0) then
         write(*,*) ' it = ',it
     end if
200   format(1x,1pE14.6,2x1pE14.6)

    end do

    close (unit=9)

1    CONTINUE

Compute the Convolution Integral:
        inquire (file='PULSE3.dat',exist=ex)
        if (ex) then
         open(11,file='PULSE3.dat',status='OLD')
         rewind 11
        else
         open(11,file='PULSE3.dat',status='NEW')
        end if

        it = 0
     dT = 1d0/(2d0*fmax)
```

```
      do n=1,npts
       t = n*dT
       X(n) = pulse(t, fc,taw)
      end do

      call CONVL (X, npts, IMP, mpts, Y, ly)

      do n=1,ly
       write(11,200) n*dT, Y(n)*dT
      end do
      close (unit=11)

      stop
      E N D

      real*8 function pulse (t, fc,on_time)
      implicit real*8 (a-h,o-z)
      pi = dacos(-1d0)

      if (t .lt. on_time) then
       pulse = dsin(2d0*pi*fc*t)
      else
       pulse = 0d0
      end if

      return
      END

      subroutine CONVL (X, n, H, m, Y, ly)
*
*     Subroutine CONVL computes the convolution
*     between the sequences X and H
*     Parameters
*       X     : array containing sequence X
*       n     : length of sequence X
*       H     : array containing sequence H
*       m     : length of sequence H
*       Y     : array containing the convolution of X and H
*       ly    : length of the sequence Y
*
      real*8 X,H,Y
      dimension X(1),H(1),Y(1)

      ly = n + m - 1
      do  k = 1 , ly
       Y(k) = 0d0
      end do

      do 1 i = 1 , n
       do 1 j = 1 , m
        k = i + j - 1
1       Y(k) = Y(k) + X(i) * H(j)

      return
```

END

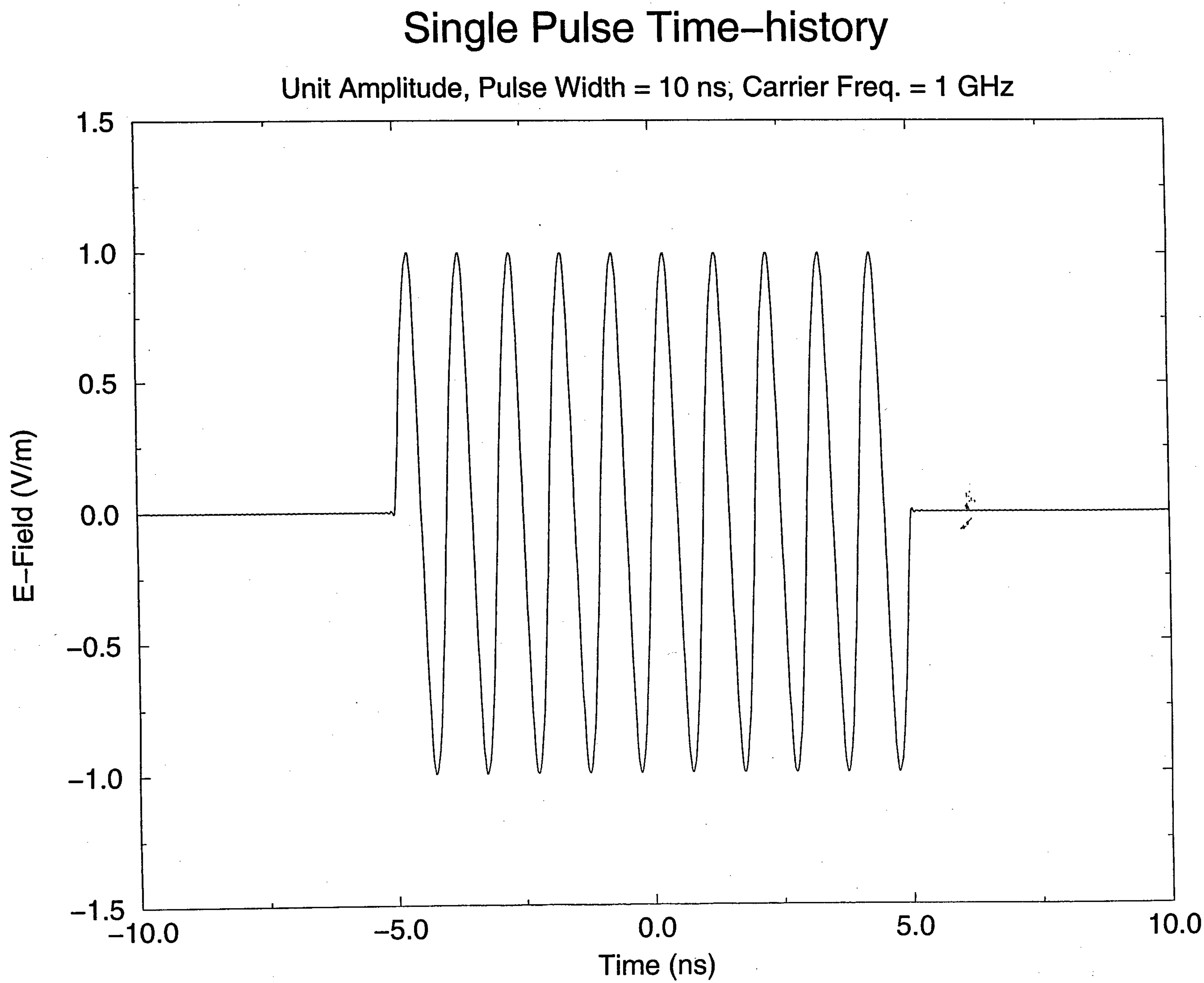


Figure 1. Sinewave pulse with 1 GHz carrier, 10 cycles

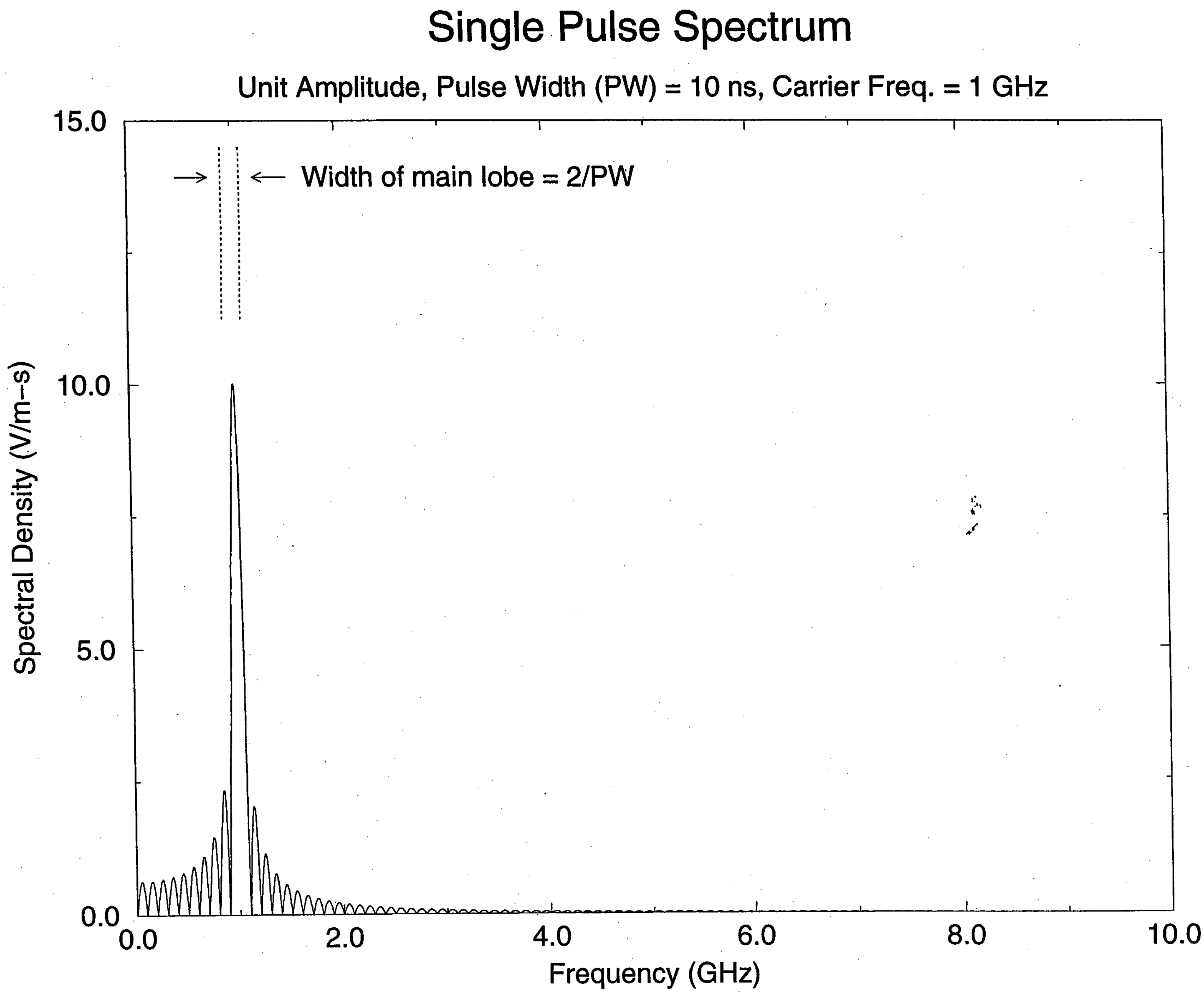


Figure 2. Spectrum of sinewave pulse with 1 GHz carrier , 10 cycles

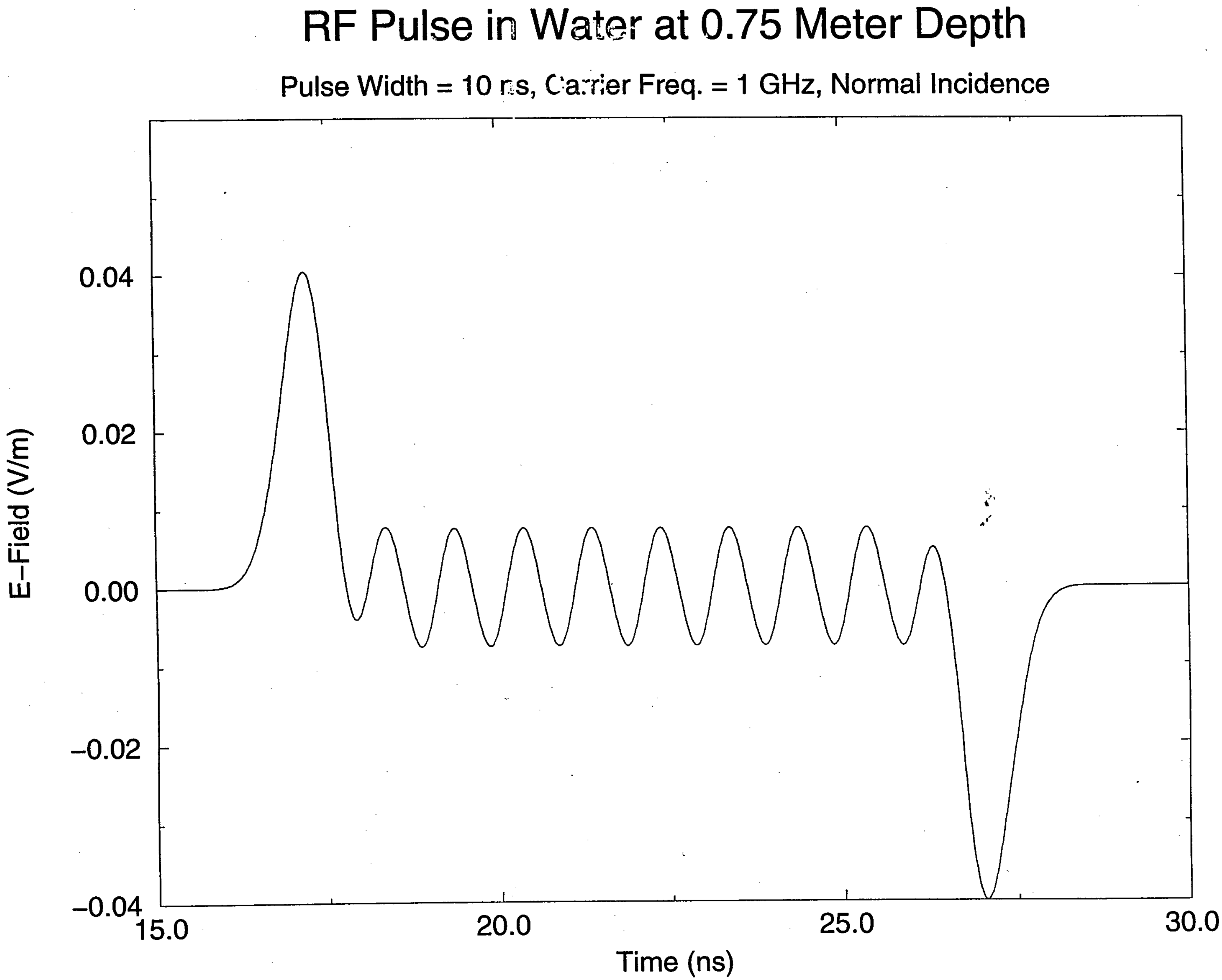


Figure 3. RF pulse in water at ¾ meter depth

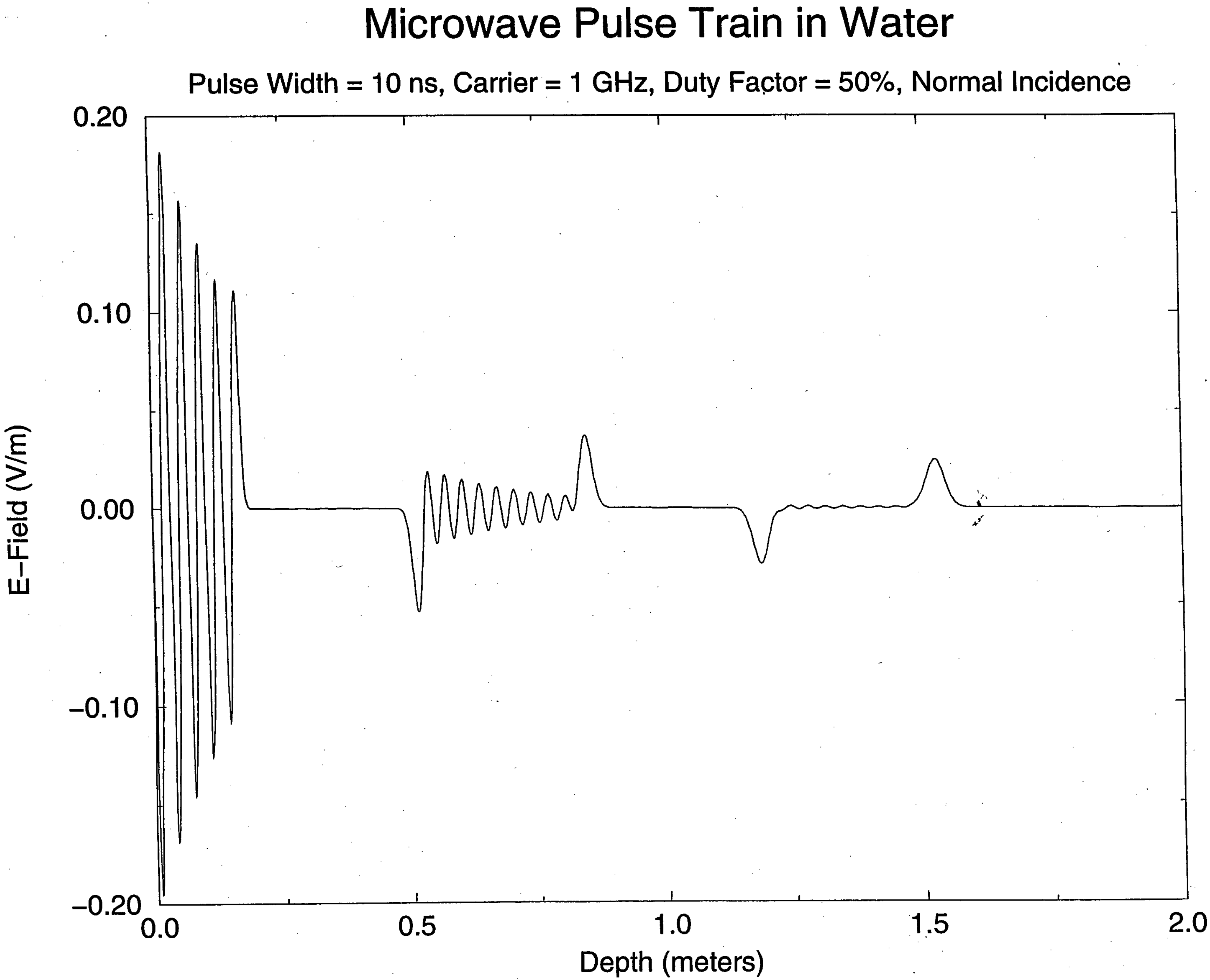


Figure 4. Sinewave pulse train in water

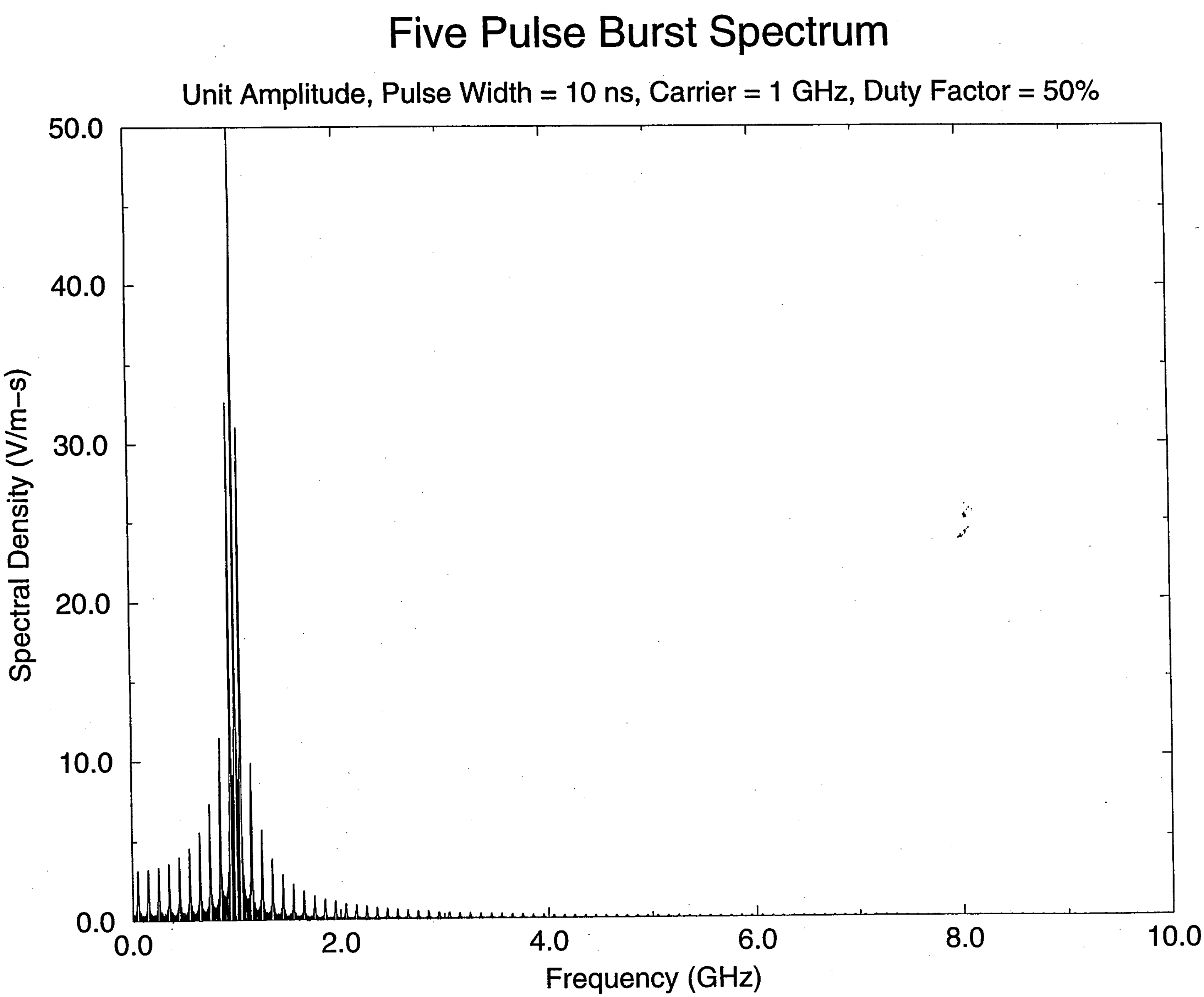
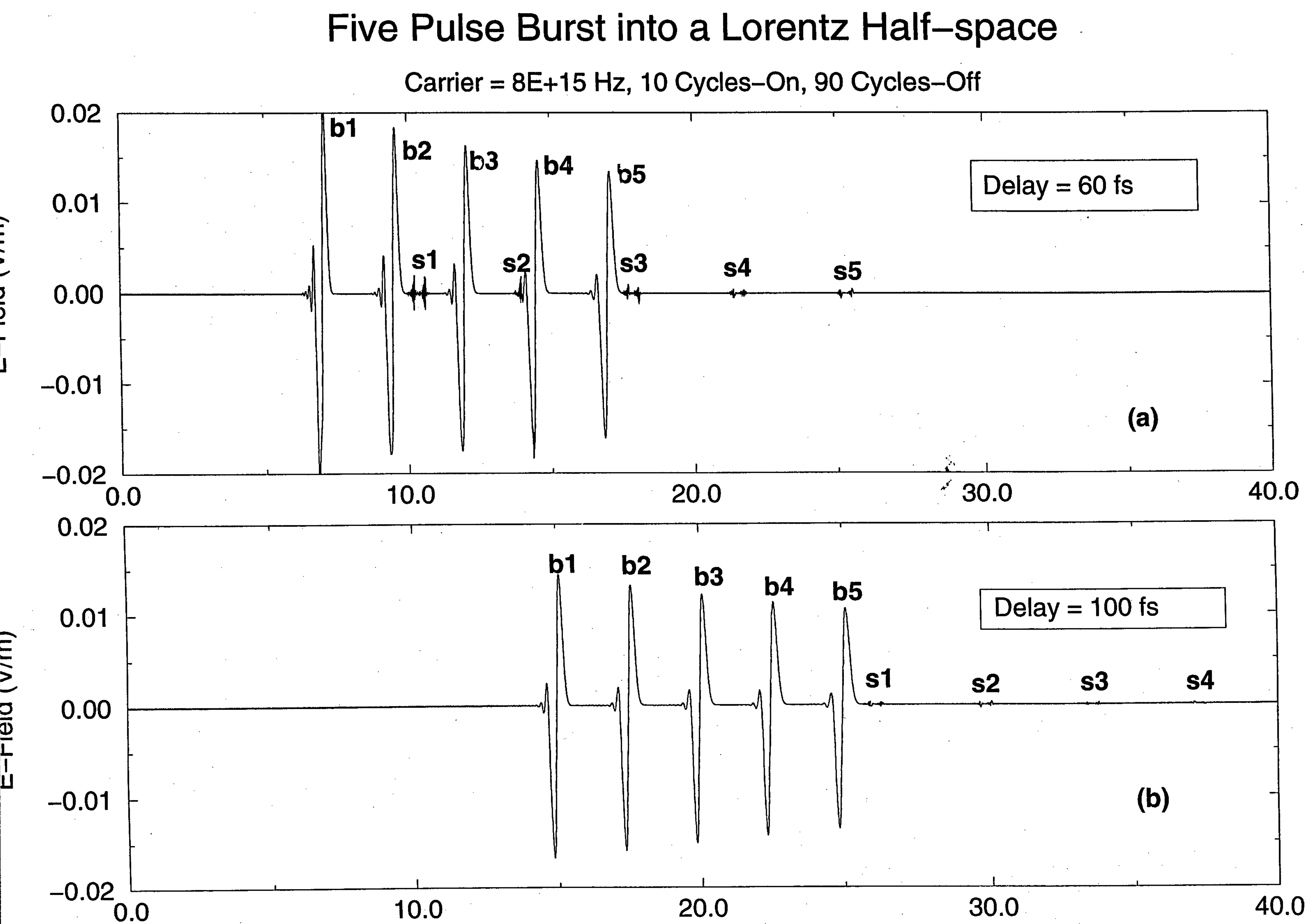


Figure 5. Spectrum of five sinewave pulses: duty factor = ½



Figures 6a & 6b. Five pulse burst into Lorenztian half-space

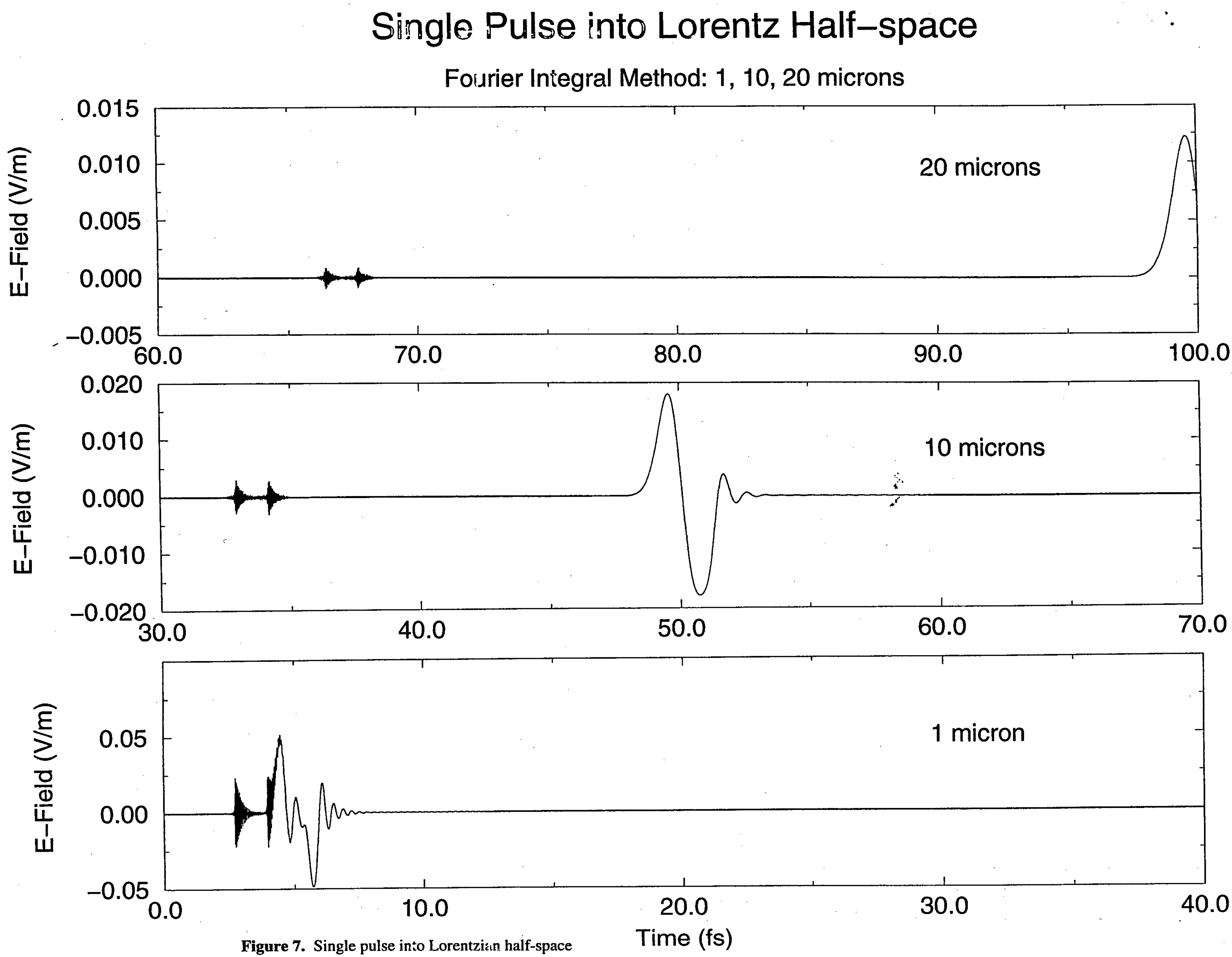


Figure 7. Single pulse into Lorentzian half-space

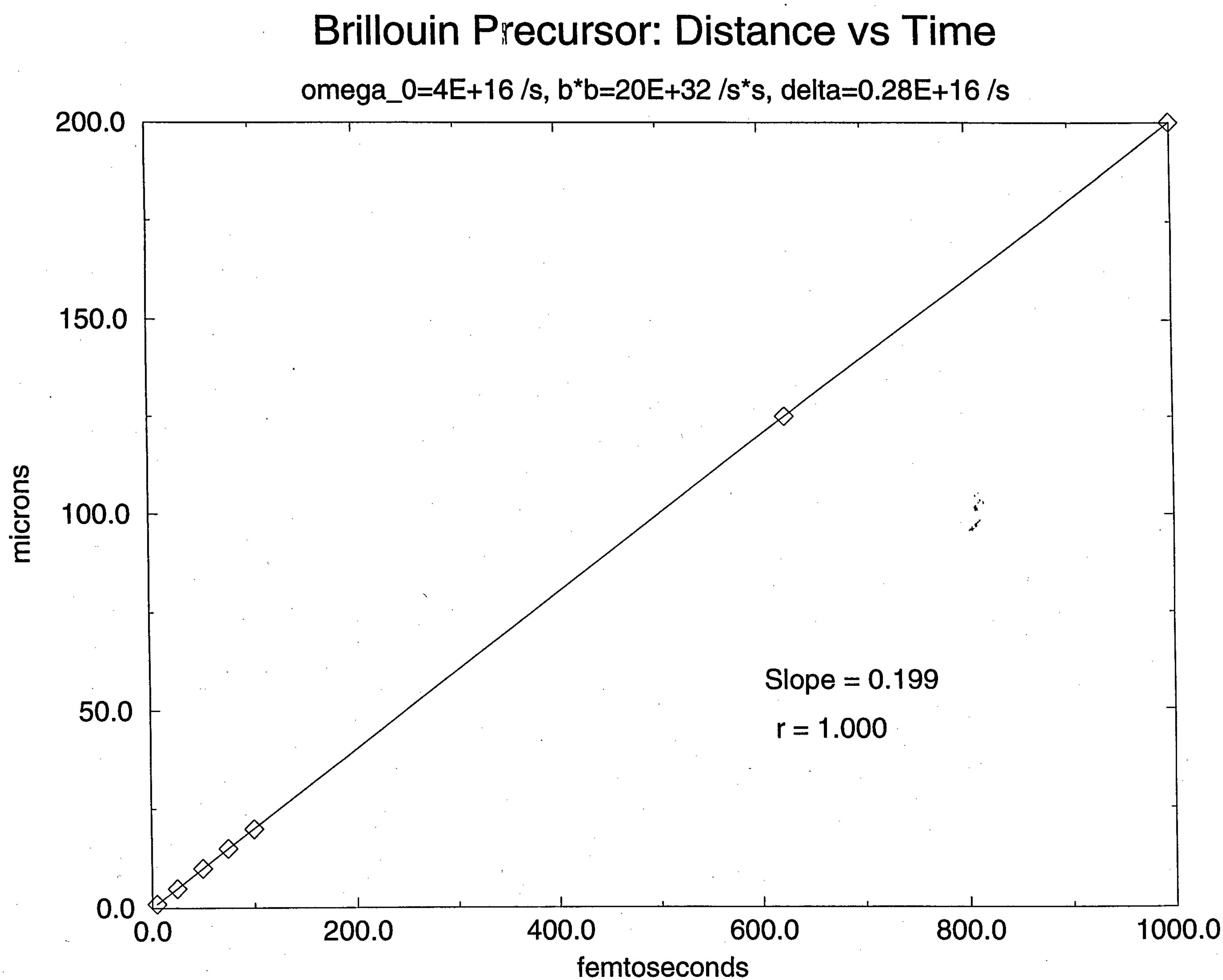


Figure 8. Brillouin precursor: distance vs. time

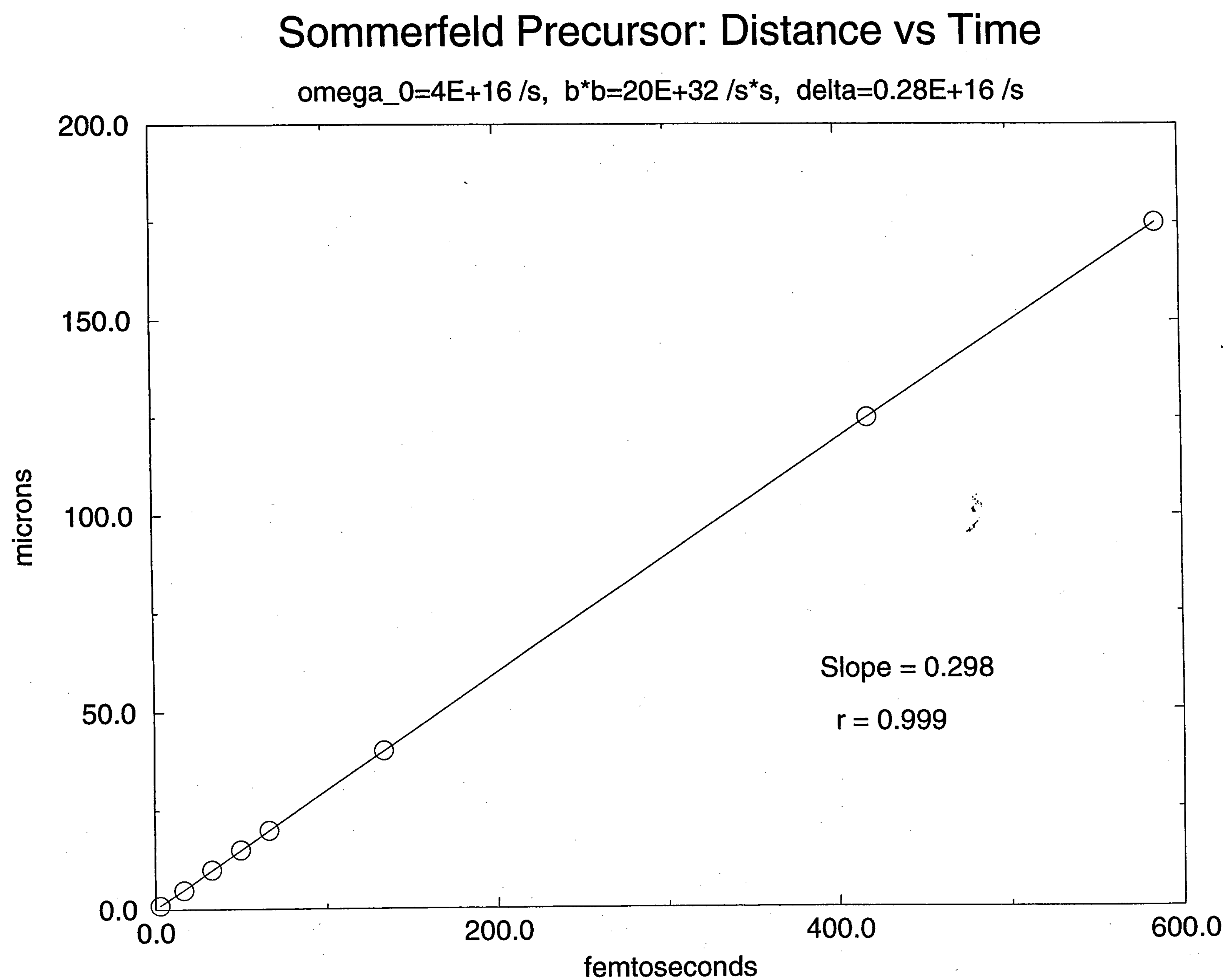


Figure 9. Sommerfeld precursor: distance vs. time

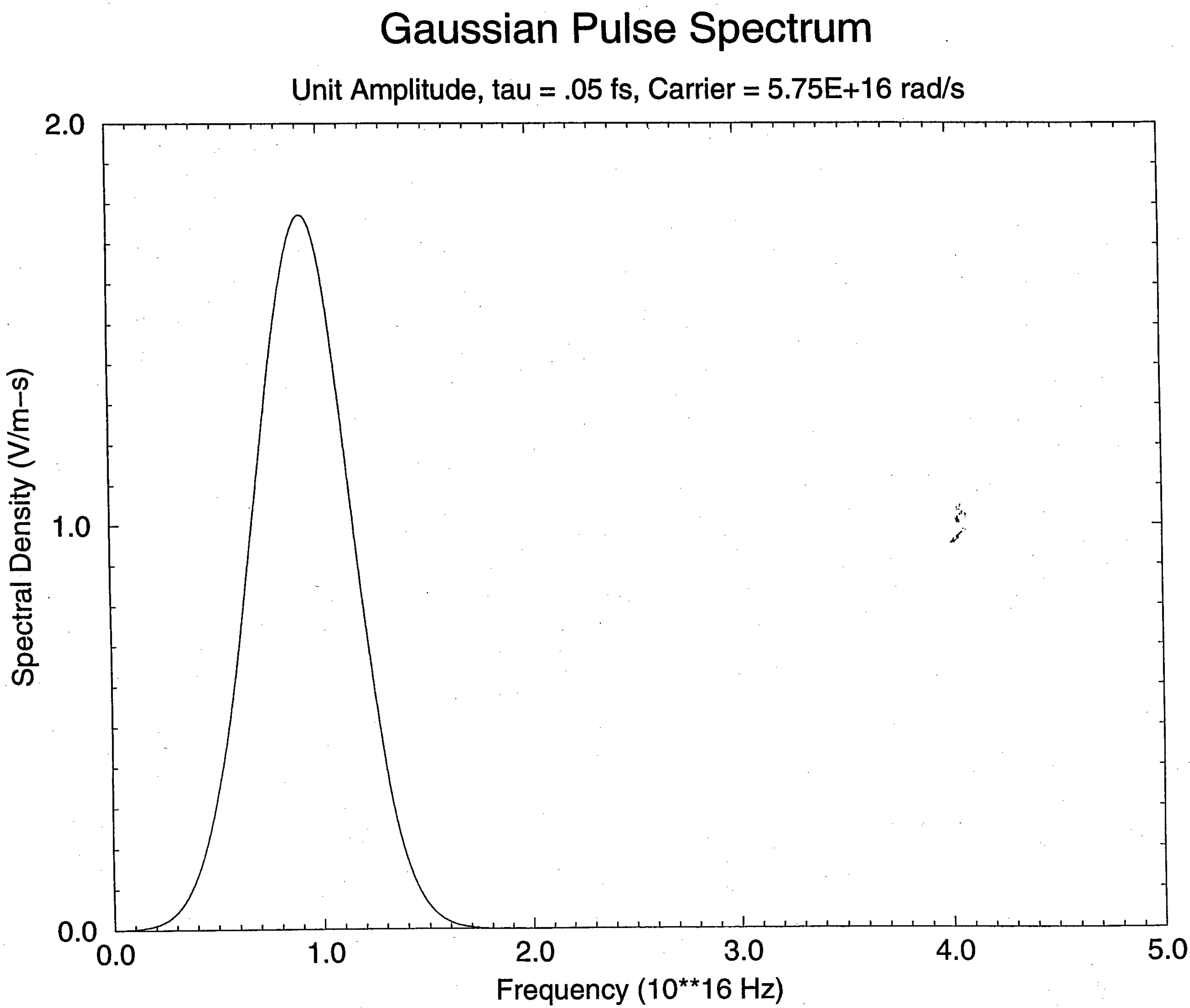


Figure 10. Spectrum of single Gaussian pulse

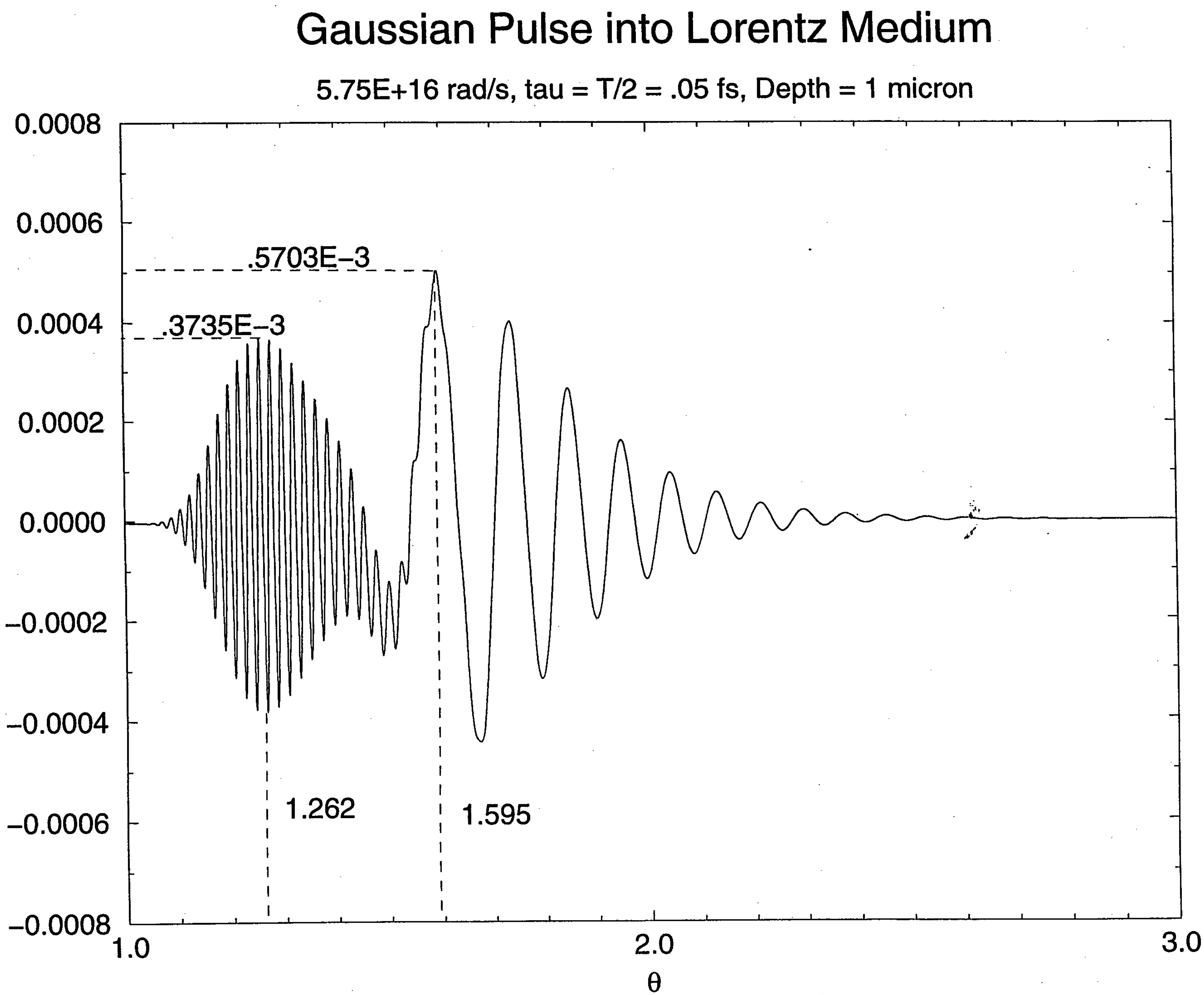


Figure 11. Gaussian pulse into Lorentzian medium

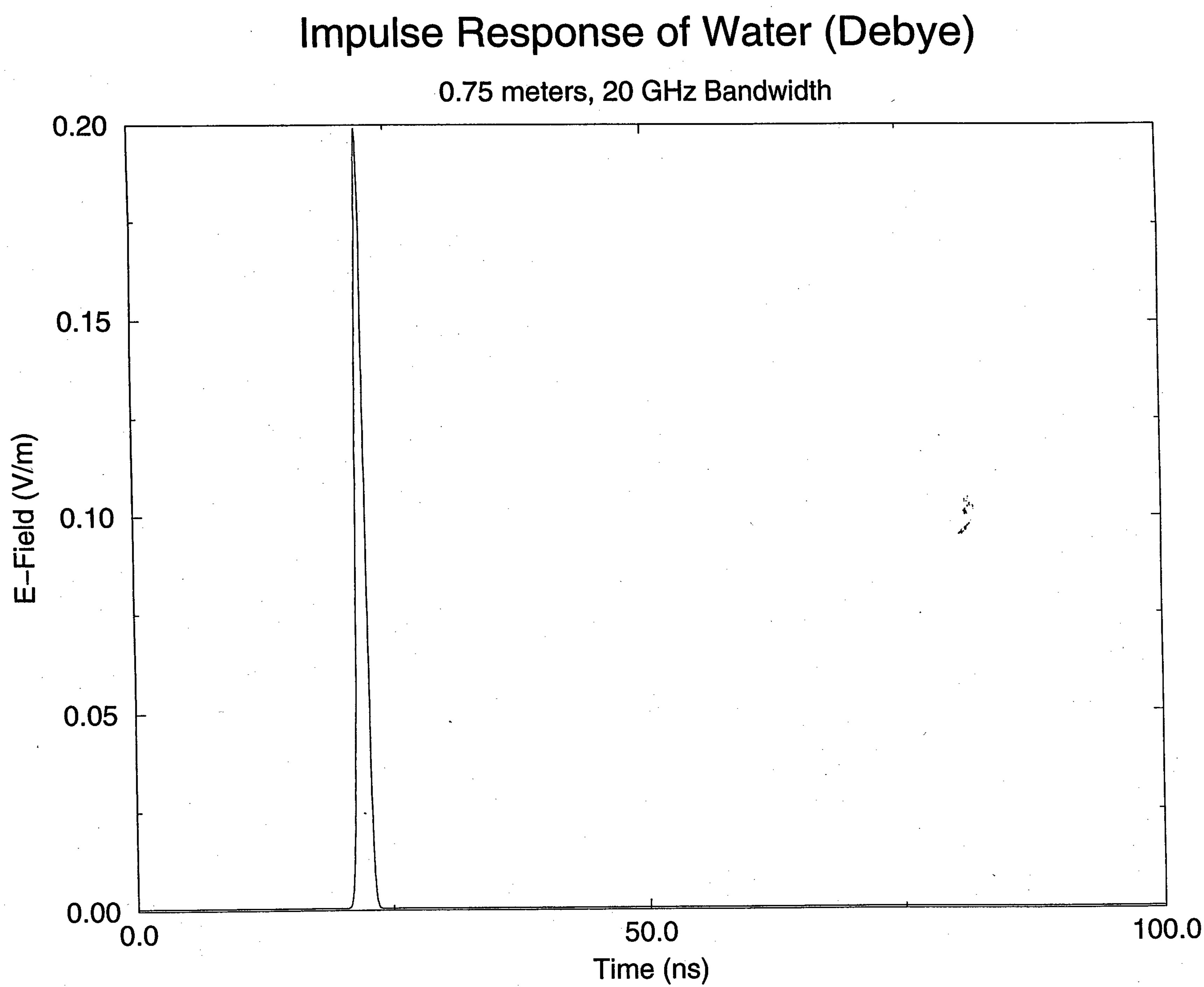


Figure 12. Impulse response of water from Debye equation

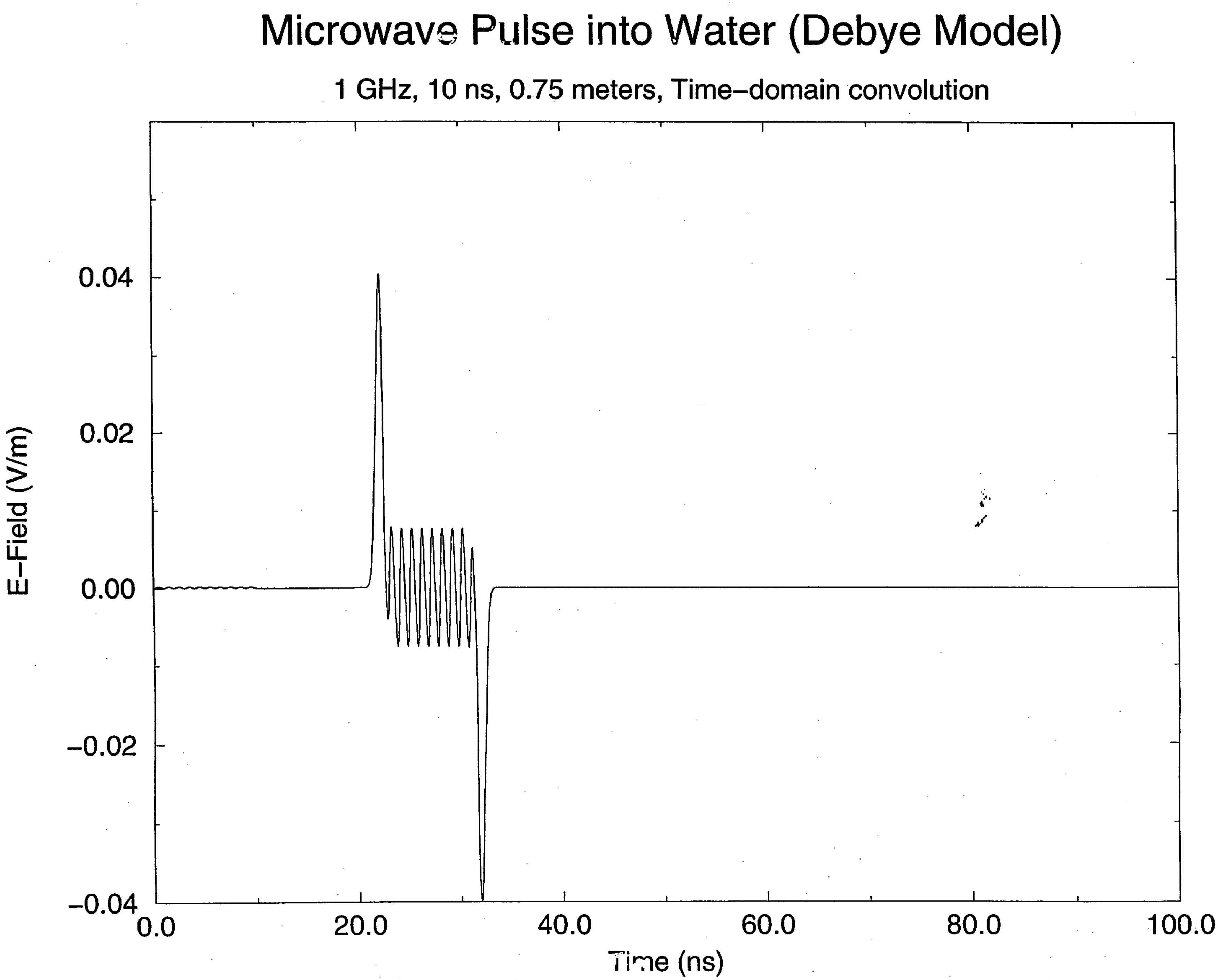


Figure 13. Sinewave pulse into water via time-domain convolution

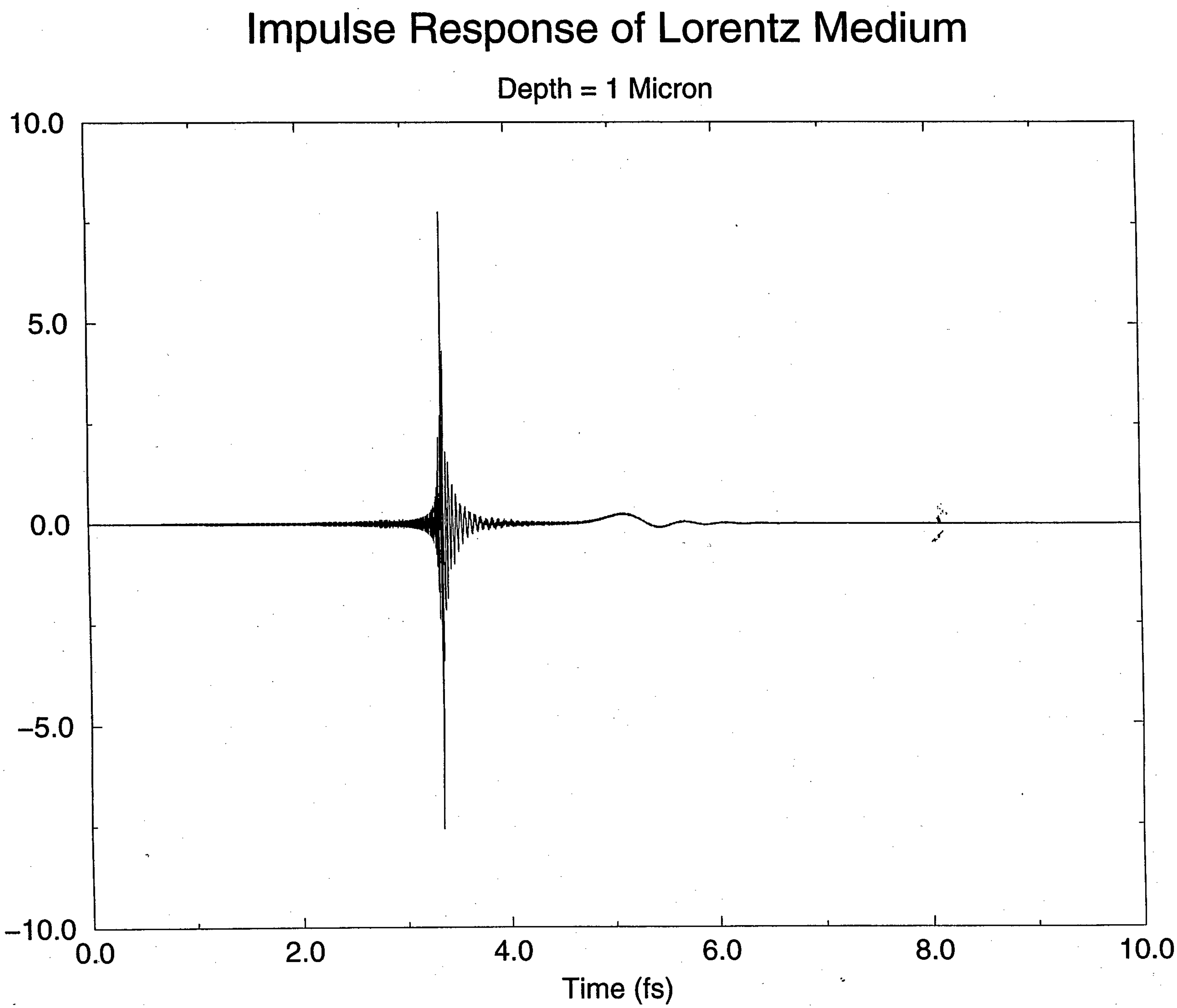


Figure 14. Impulse response of Lorentzian medium

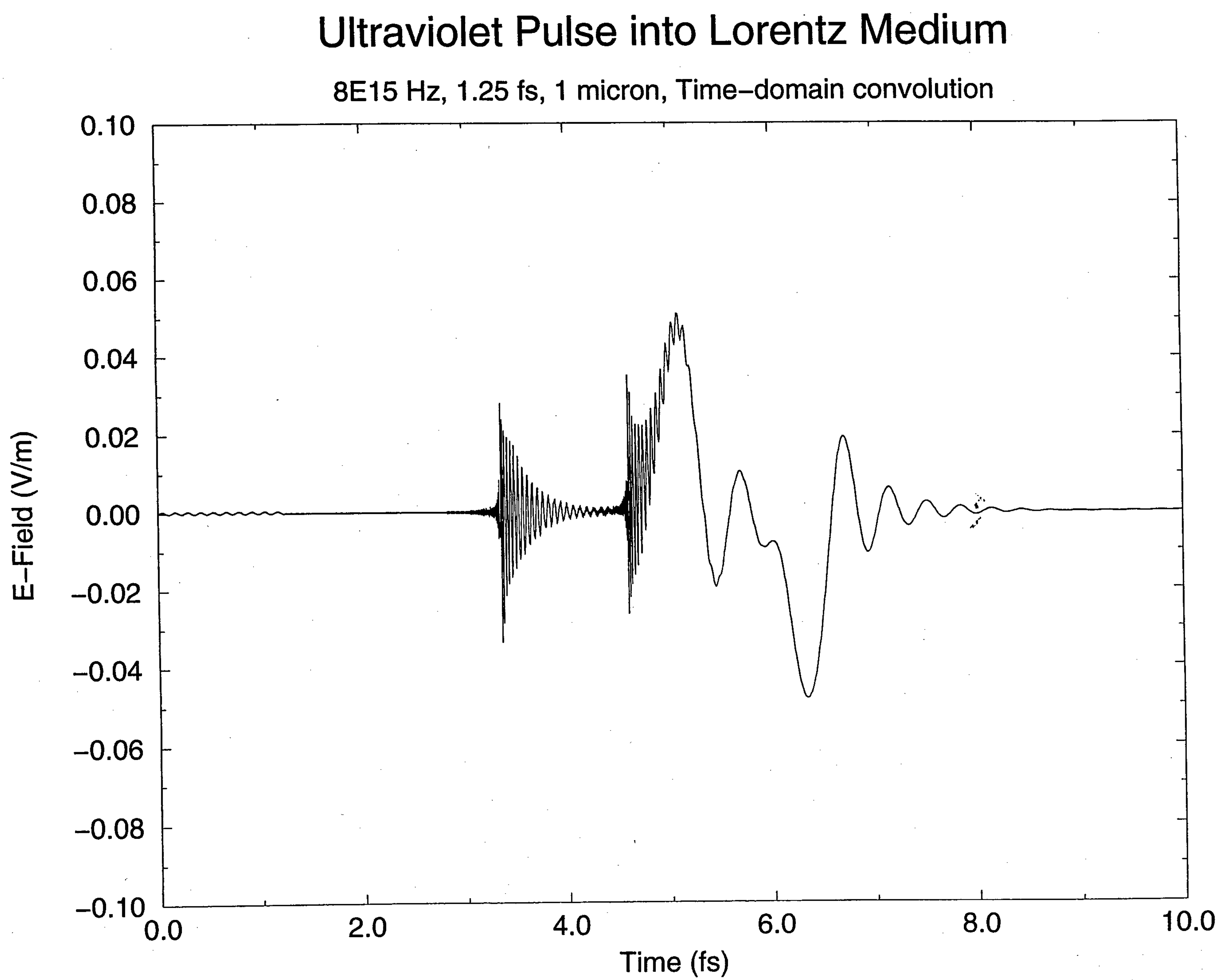


Figure 15. UV pulse into Lorentzian medium via time-domain convolution

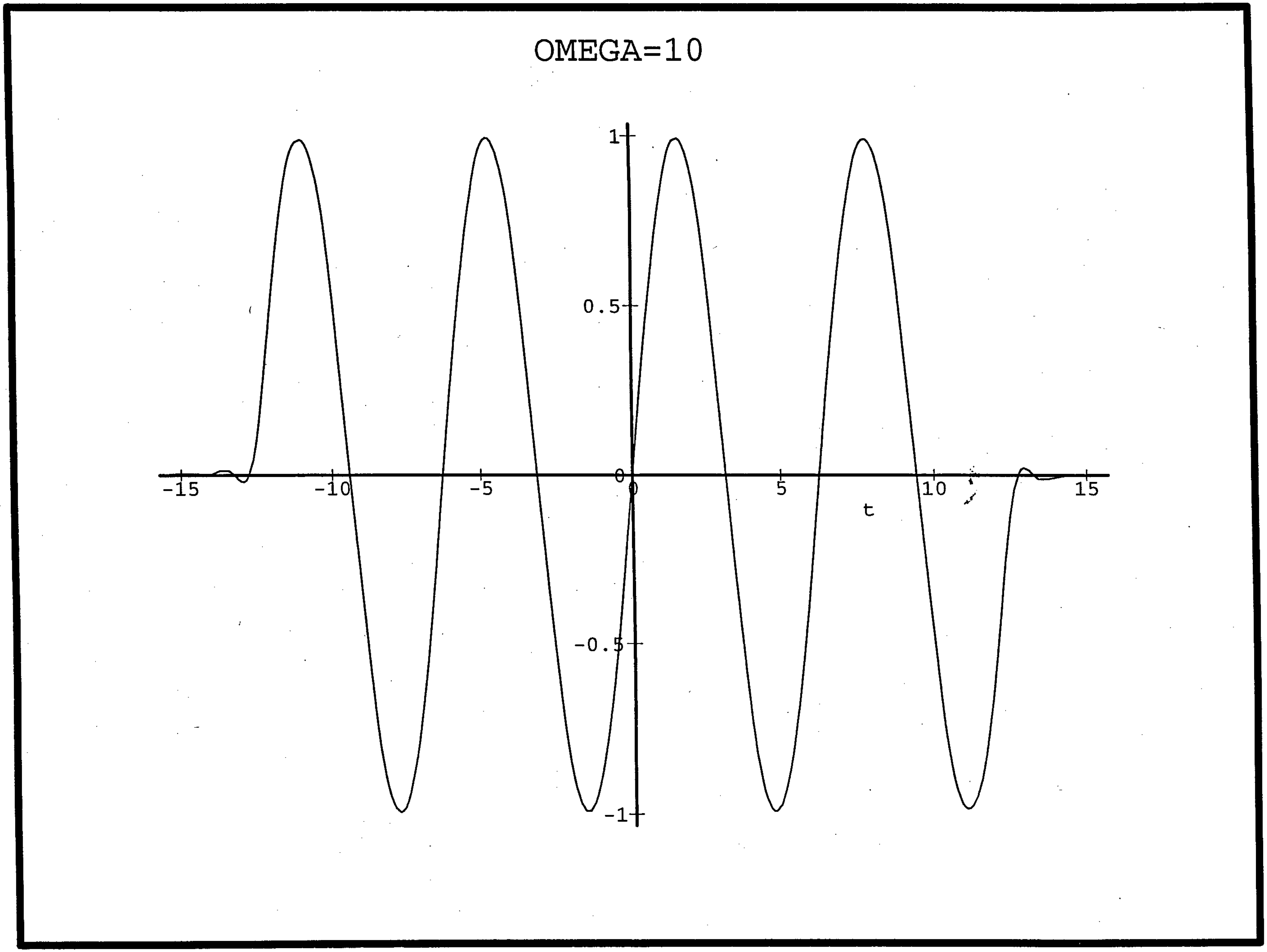


Figure 16. Four cycle pulse reconstructed using 10 rad/s integration bandwidth

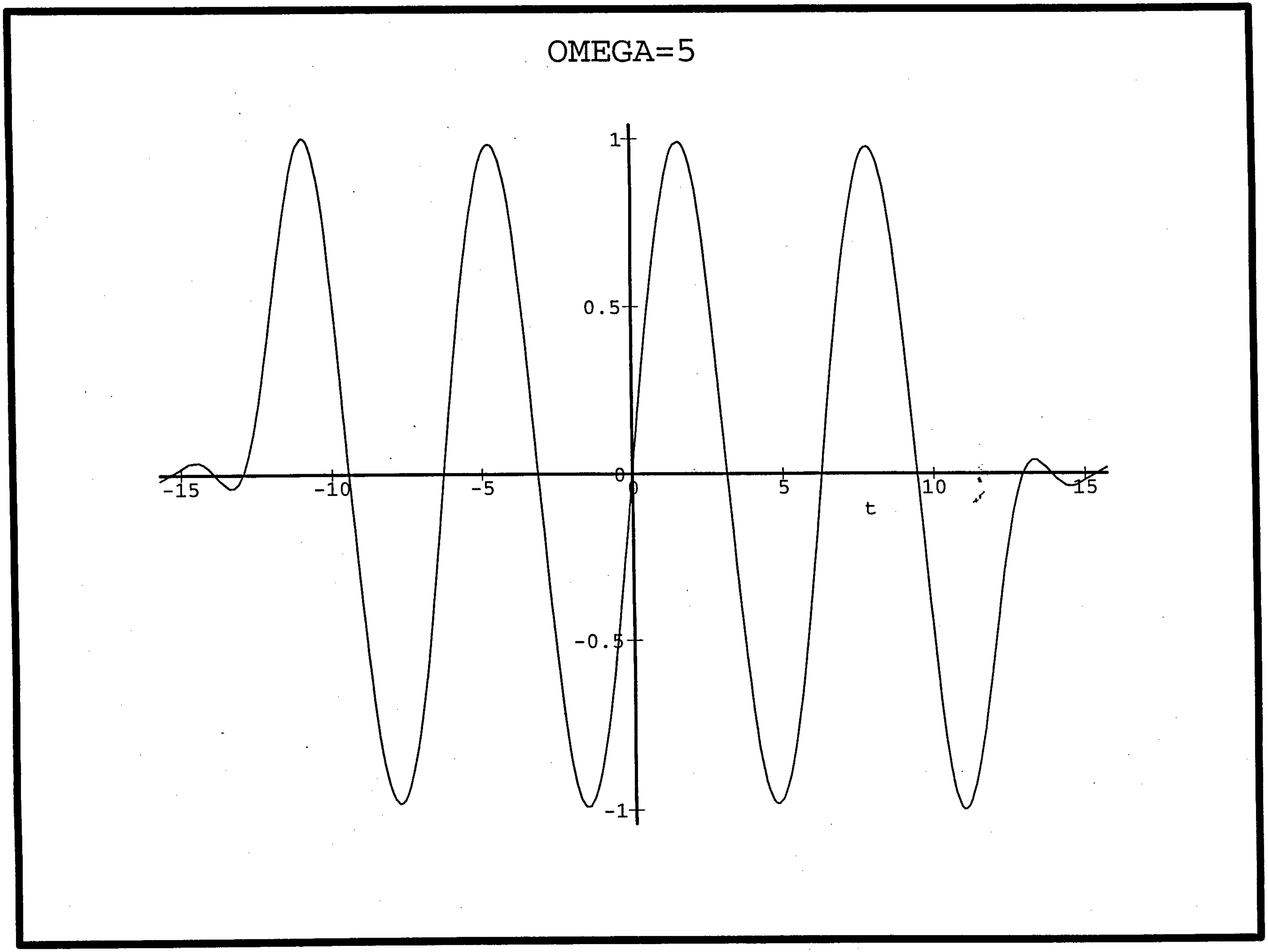


Figure 17. Four cycle pulse reconstructed using 5 rad/s integration bandwidth

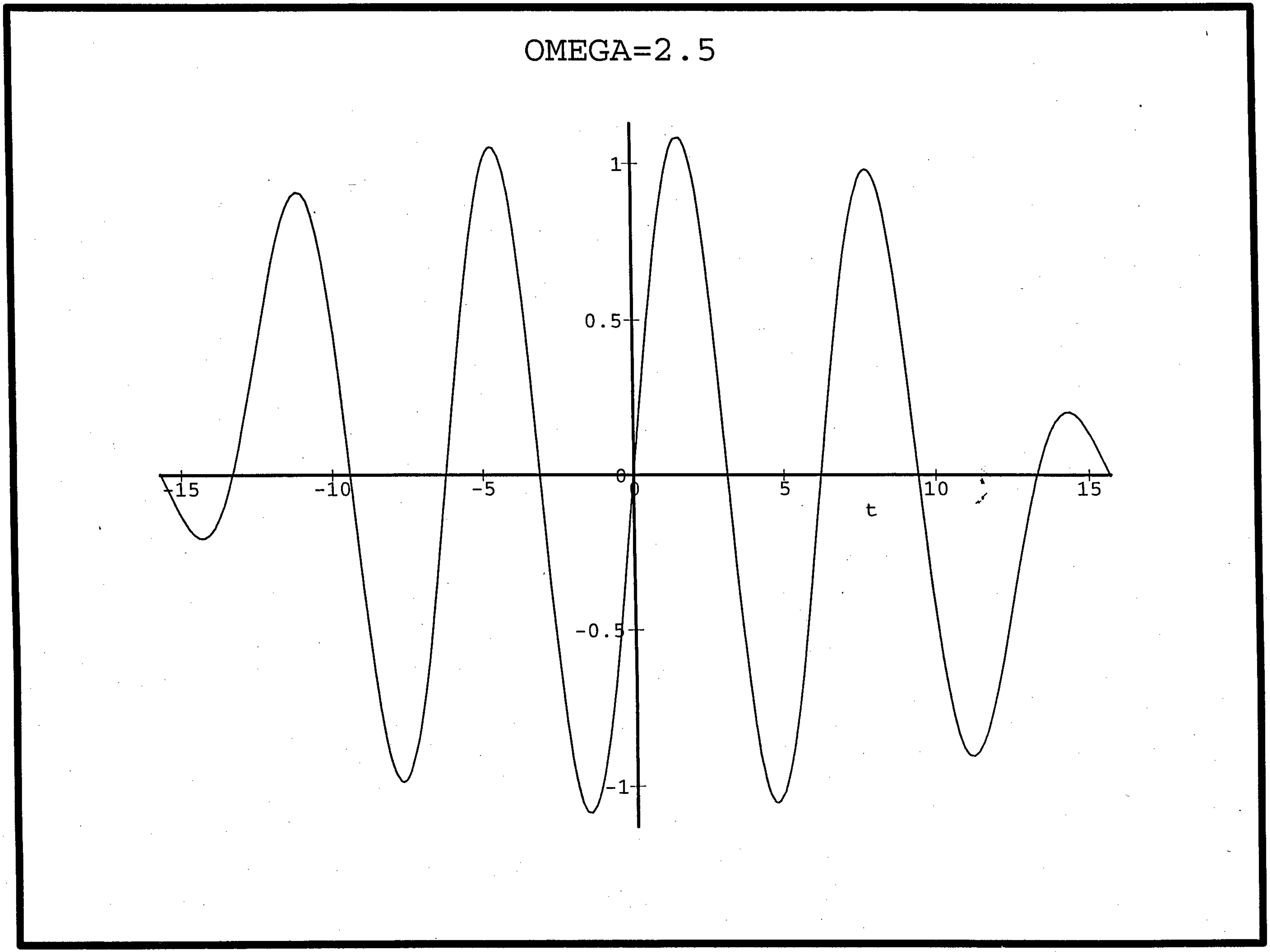


Figure 18. Four cycle pulse reconstructed using 2.5 rad/s integration bandwidth

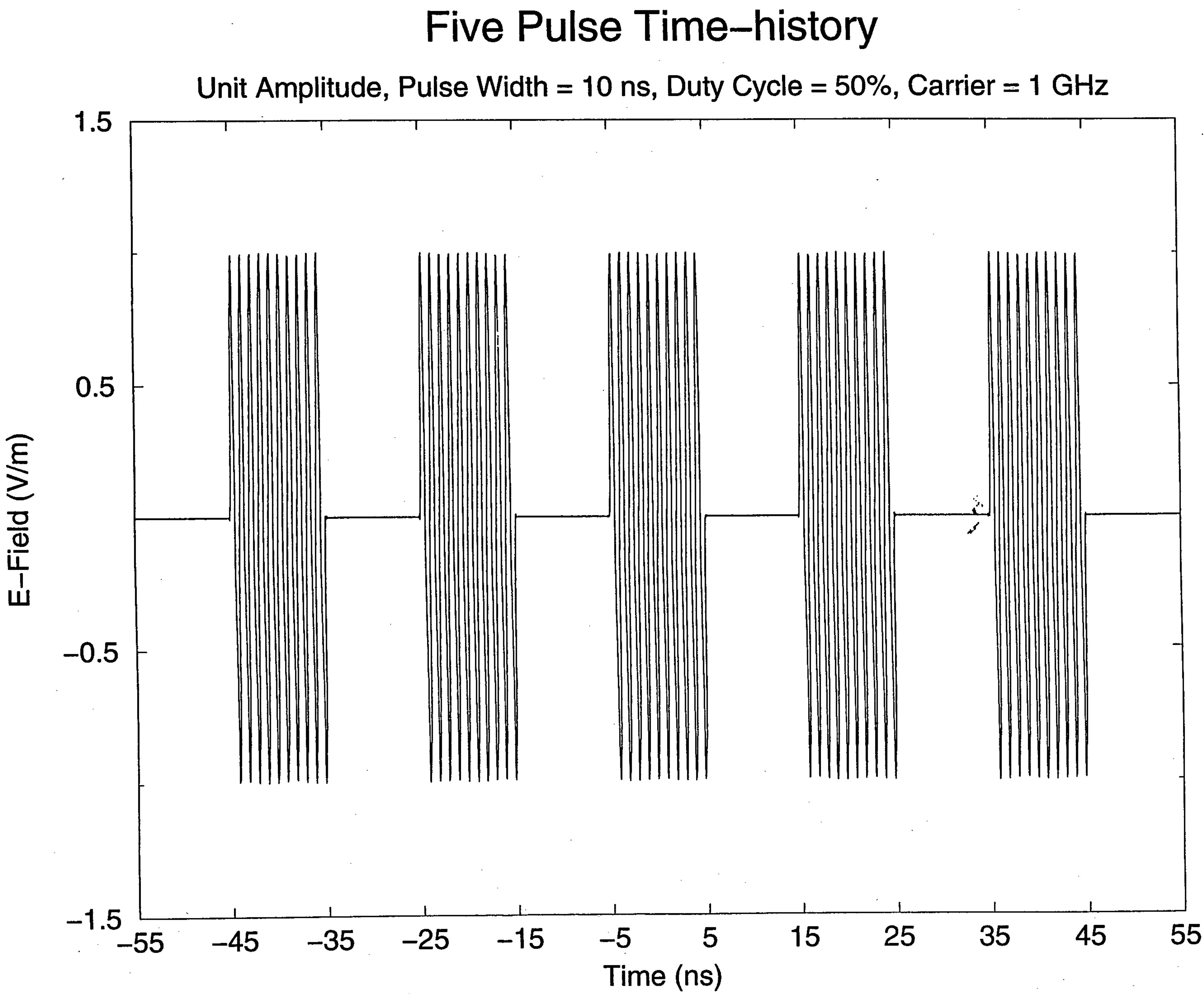


Figure A1. Five pulse wave train